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Orthogonal Polynomials of Types A and B and Related Calogero Models

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Abstract

There are examples of Calogero-Sutherland models associated to the Weyl groups of type A and B. When exchange terms are added to the Hamiltonians the systems have non-symmetric eigenfunctions, which can be expressed as products of the ground state with members of a family of orthogonal polynomials. These polynomials can be defined and studied by using the differential-difference operators introduced by the author in Trans. Amer. Math. Soc. 1989 (311), 167-183. After a description of known results, particularly from the works of Baker and Forrester, and Sahi; there is a study of polynomials which are invariant or alternating for parabolic subgroups of the symmetric group. The detailed analysis depends on using two bases of polynomials, one of which transforms monomially under group actions and the other one is orthogonal. There are formulas for norms and point-evaluations which are simplifications of those of Sahi. For any parabolic subgroup of the symmetric group there is a skew operator on polynomials which leads to evaluation at $(1, 1, \ldots, 1)$ of the quotient of the unique skew polynomial in a given irreducible subspace by the minimum alternating polynomial, analogously to a Weyl character formula. The last section concerns orthogonal polynomials for the type B Weyl group with an emphasis on the Hermite-type polynomials. These can be expressed by using the generalized binomial coefficients. A complete basis of eigenfunctions of Yamamoto's B_N spin Calogero model is obtained by multiplying these polynomials by the ground state.

1 Introduction

A Calogero-Sutherland model is an exactly solvable quantum many-body system in one dimension. There are examples associated to the Weyl groups of type A and B, and by the addition of exchange (reflection) terms the Hamiltonians have non-symmetric eigenfunctions. In the two situations described here, the eigenfunctions are polynomials times the ground state.

The first example consists of N particles on a circle, with particle j being at angle θ_j, $0 \le \theta_j < 2\pi$, parameter $k > 0$; the Hamiltonian is

(1.1) $$\mathcal{H}_1 := -\sum_{i=1}^{N} \left(\frac{\partial}{\partial \theta_i}\right)^2 + \frac{k}{2} \sum_{1 \le i < j \le N} \frac{k - (ij)}{\sin^2(\frac{1}{2}(\theta_i - \theta_j))},$$

where (ij) denotes the transposition ("exchange") $\theta_i \longleftrightarrow \theta_j$.

Under the transformation $x_s = \exp(\theta_s\sqrt{-1})$,

$$\mathcal{H}_1 = \sum_{i=1}^{N}\left(x_i\frac{\partial}{\partial x_i}\right)^2 + 2k\sum_{1\le i<j\le N}\frac{x_ix_j}{(x_i-x_j)^2}((ij)-k). \tag{1.2}$$

The orthogonal polynomials associated to $\mathcal{H}_1$ are called the non-symmetric Jack polynomials; see Baker and Forrester [BF1], Lapointe and Vinet [LV2]. In Section 2 this Hamiltonian will be further described.

The second example to be studied is the B-type spin Calogero model of Yamamoto [[Y], [YT]]; parameters k, k_1:

$$\begin{aligned}\mathcal{H}_2 \;&=\; -\sum_{i=1}^{N}\left(\frac{\partial}{\partial x_i}\right)^2 + \frac{1}{4}\sum_{i=1}^{N}x_i^2 + \sum_{i=1}^{N}\frac{k_1(k_1-\sigma_i)}{x_i^2}\\ &\quad + 2k\sum_{1\le i<j\le N}\left\{\frac{k-\sigma_{ij}}{(x_i-x_j)^2}+\frac{k-\tau_{ij}}{(x_i+x_j)^2}\right\},\end{aligned} \tag{1.3}$$

where $\sigma_i, \sigma_{ij}, \tau_{ij}$ are the reflections in the hyperoctahedral group W_N, defined by $x\sigma_i = (x_1,\ldots,-\overset{i}{x_i},\ldots,x_N)$, $x\sigma_{ij} = (\ldots,\overset{i}{x_j},\ldots,\overset{j}{x_i},\ldots)$, $x\tau_{ij} = (\ldots,-\overset{i}{x_j},\ldots,-\overset{j}{x_i},\ldots)$. The coefficient $\frac{1}{4}$ in $\mathcal{H}_2$ is a coupling constant; it can be changed by rescaling x; this choice is to use the weight function $\exp(-|x|^2/2)$, as will be seen later. In Section 5 is the description of a complete orthogonal system of eigenfunctions of $\mathcal{H}_2$, consisting of polynomials times the ground state

$$\prod_{1\le i<j\le N}|x_i^2-x_j^2|^k\prod_{i=1}^{N}|x_i|^{k_1}\exp(-|x|^2/4).$$

The technical foundation of this paper is the algebra of differential-difference ("Dunkl") operators associated to a reflection group [D1]. In Section 2, there is an outline of known results for the type A case, namely the symmetric group S_N acting on $\mathbb{R}^N$ by permutation of coordinates. This section includes a discussion of inner products and self-adjoint operators, and orthogonal decompositions.

Section 3 is concerned with polynomials and operators invariant under parabolic subgroups of S_N; these are subgroups which leave intervals $\{1,2,3,\ldots,\ell_1\}, \{\ell_1{+}1,\ldots,\ell_1{+}\ell_2\},\ldots$ invariant. Formulas for the norms of invariant polynomials are obtained.

In Section 4, the alternating or skew polynomials and operators are examined. There is the construction of an important operator associated to any interval $\{\ell+1,\ell+2,\ldots,\ell+m\}$ which is skew for the associated symmetric group, and which commutes with the appropriately transformed version of the Hamiltonian $\mathcal{H}_1$. Any polynomial which is skew-symmetric for a parabolic (Young) subgroup of S_N is divisible by an appropriate minimal alternating polynomial (a product of discriminants). The skew operator is used to evaluate the ratio at $x = (1,1,\ldots,1)$, a generalization of the Weyl dimension formula.

Section 5 addresses the type B situation and shows how the type A polynomials can be used to build type B Hermite polynomials, the eigenfunctions of the transformed $\mathcal{H}_2$. This results in a complete set of eigenfunctions with arbitrary parity, that is for any subset $A \subset \{1, 2, \dots, N\}$ there are eigenfunctions which are odd in x_i, $i \in A$ and even in x_i, $i \notin A$. Previously, only the cases of all even or all odd parity were studied, the so-called generalized Laguerre polynomials.

Notations Used Throughout

- $Z_+ = \{0, 1, 2, 3, \dots\}$, $\mathcal{N}_N = Z_+^N$, the set of compositions;
- $\mathcal{N}_N^P$ is the set of partitions with no more than N nonzero parts; $\mathcal{N}_N^P = \{\lambda \in \mathcal{N}_N : \ \lambda_1 \geq \lambda_2 \geq \lambda_3 \cdots \geq \lambda_N \geq 0\}$;
- for $\alpha \in \mathcal{N}_N$, α^+ denotes the sorting of α to a partition; the permutation of α lying in $\mathcal{N}_N^P$;
- for $\alpha, \beta \in \mathcal{N}_N$, the dominance order is defined by $\alpha \succeq \beta$ if and only if $\sum_{i=1}^j \alpha_i \geq \sum_{i=1}^j \beta_i$ for $1 \leq j \leq N$; and $\alpha \succ \beta$ means $\alpha \succeq \beta$ and $\alpha \neq \beta$;
- for $w \in S_N$, the symmetric group, and $x \in \mathbb{R}^N$, let $xw \in \mathbb{R}^N$ be defined by $(xw)_i = x_{w(i)}, \ 1 \leq i \leq N$, and $\mathrm{sgn}(w)$ denotes the sign of w;
- for a function f on $\mathbb{R}^N$, let $(wf)(x) = f(xw), \ x \in \mathbb{R}^N$;
- for $\alpha \in \mathcal{N}_N$, let $x^\alpha := \prod_{i=1}^N x_i^{\alpha_i}$, then $w(x^\alpha) = x^{w\alpha}$, where $(w\alpha)_i = \alpha_{w^{-1}(i)}, \ 1 \leq i \leq N$, $w \in S_N$;
- an interval $[\ell + 1, \ell + m] := \{j \in \mathbb{Z} : \ \ell + 1 \leq j \leq \ell + m\}$;
- for an interval I, let $S_I = \{w \in S_N : \ i \notin I \text{ implies } w(i) = i\}$ (that is, S_I is isomorphic to the symmetric group of I);
- for an interval I, let σ_I be the longest element in S_I, that is, $\sigma_I(i) = 2\ell + m + 1 - i, \ i \in I = [\ell + 1, \ell + m]$;
- for an interval I, the associated alternating polynomial is $a_I(x) = \prod\{x_i - x_j : \ i < j \text{ and } i, j \in I\}$;
- for $\alpha \in \mathcal{N}_N, \ |\alpha| := \sum_{i=1}^N \alpha_i, \ \alpha! := \prod_{i=1}^N (\alpha_i!)$;
- for a set A, $\#A$ is the cardinality;
- for $\lambda \in \mathcal{N}_N^P$, and $t \in \mathbb{R}$, the hook length product is $h(\lambda, t) := \prod_{i=1}^N \prod_{j=1}^{\lambda_i} (\lambda_i - j + t + k(\#\{s : s > i \text{ and } j \leq \lambda_s \leq \lambda_i\})$;
- for $\alpha \in \mathcal{N}_N, \ 1 \leq i \leq N, \ \kappa_i(\alpha) = Nk - k(\#\{s : \ \alpha_s > \alpha_i\} + \#\{s : \ s < i \text{ and } \alpha_s = \alpha_i\}) + \alpha_i + 1$ (a frequently used eigenvalue associated to α);

- for an interval I, $\alpha \in \mathcal{N}_N$ satisfies condition $(\geq, I)$ respectively $(>, I)$ if $i, j \in I$ and $i < j$ implies $\alpha_i \geq \alpha_j$, respectively $\alpha_i > \alpha_j$;

- for two linear operators A, B the commutator is $[A, B] := AB - BA$;

- for $t \in \mathbb{R},\ m \in \mathbb{Z}_+$, the shifted factorial is $(t)_m = \prod_{i=1}^m (t + i - 1)$; for $\lambda \in \mathcal{N}_N^P$ (and implicit parameter k) the generalized shifted factorial is $(t)_\lambda := \prod_{i=1}^N (t - (i-1)k)_{\lambda_i}$;

- $1^N = (1, 1, \ldots, 1) \in \mathbb{R}^N$.

2 Background

We review facts about the non-symmetric Jack polynomials expressed in two different bases, the relation to the operators introduced by Cherednik, and the inductive calculation of norm formulas by use of adjacent transpositions.

The symmetric group S_N acts on $\mathbb{R}^N$ by permutation of coordinates and thus extends to an action on functions

$$wf(x) := f(xw),\ x \in \mathbb{R}^N,\ w \in S_N.$$

For a parameter $k \geq 0$, the type A Dunkl operators are defined by

$$T_i = \frac{\partial}{\partial x_i} + k \sum_{j \neq i} \frac{1}{x_i - x_j}(1 - (ij)),\ 1 \leq i \leq N. \tag{2.1}$$

For each partition $\lambda \in \mathcal{N}_N^P$ (henceforth partitions will be assumed to have no more than N parts), there is a space E_λ of polynomials, invariant and irreducible under the algebra generated by $\{T_i x_i :\ 1 \leq i \leq N\}$ and $\{w :\ w \in S_N\}$. This algebra can be considered as a subalgebra of the degenerate double affine Hecke algebra of type A (the latter acts on Laurent series and also contains the multiplications by x_i^{-1}, $1 \leq i \leq N$, Cherednik [C], Kakei [K3]).

There is a simple relationship between the rational and trigonometric differential-difference operators of type A; indeed, let $x_i = e^{y_i}$, $1 \leq i \leq n$; and suppose f is a linear combination of $\{e^{y_i}, e^{-y_i} :\ 1 \leq i \leq N\}$, then

$$(T_i x_i) f = (1 + (N-1)k) f + \frac{\partial f}{\partial y_i} + k \sum_{j \neq \ell} \frac{1}{e^{y_i - y_j} - 1}\ (f - (ij)f).$$

However, there is no corresponding relationship for type B (the weight function for rational type B on the N-torus only involves one parameter).

For $\alpha \in \mathcal{N}_N$,

$$\begin{aligned} T_i x_i x^\alpha &= (Nk - k\#\{j :\ \alpha_j > \alpha_i\} + \alpha_i + 1) x^\alpha \\ &\quad - k \sum_{\alpha_j > \alpha_i + 1} \sum_{s=0}^{\alpha_j - \alpha_i - 2} x^\alpha x_i^{s+1} x_j^{-s-1} \end{aligned}$$

$$+ k \sum_{\alpha_j \leq \alpha_i} \sum_{s=1}^{\alpha_i - \alpha_j} x^\alpha x_i^{-s} x_j^s;$$

every x^β in the sums satisfies $\beta^+ \prec \alpha^+$ except the cases $s = \alpha_i - \alpha_j > 0$ which produce $k(ij)x^\alpha$. When $j > i$ and $\alpha_i > \alpha_j$, $(ij)\alpha \prec \alpha$.

Thus the operator $U_i := T_i x_i - k \sum_{j<i} (ij)$ satisfies the triangularity property

$$U_i x^\alpha = \kappa_i(\alpha) x^\alpha + \sum \{A(\beta, \alpha) x^\beta : |\beta| = |\alpha|, \beta^+ \prec \alpha^+ \text{ or } \alpha^+ = \beta^+ \text{ and } \alpha \succ \beta\}.$$

The type-A Cherednik operator ξ_i (as in [BF3]) is defined by

$$\xi_i = \left(\frac{1}{k}\right) x_i \frac{\partial}{\partial x_i} + x_i \sum_{j<i} \frac{1 - (ij)}{x_i - x_j} + \sum_{j>i} \frac{x_j (1 - (ij))}{x_i - x_j} + 1 - i,$$

and satisfies $\xi_i = \left(\frac{1}{k}\right)(U_i - (k(N-1)+1))$. The set $\{U_i : i = 1, \ldots, N\}$ is commutative (more details below), thus there is a basis (for polynomials) of simultaneous eigenfunctions, called non-symmetric Jack polynomials. The notation $E_\alpha(x; 1/k)$ is used for the normalization having one as leading coefficient (of x^α); that is,

$$E_\alpha(x; 1/k) = x^\alpha + \sum \{A'(\beta, \alpha) x^\beta : |\beta| = |\alpha|, \beta^+ \prec \alpha^+ \text{ or } \beta^+ = \alpha^+ \text{ and } \beta \prec \alpha\}$$

(and the coefficients $A'(\beta, \alpha)$ depend on k and N).

In the present paper, we use the dual basis $\{p_\alpha : \alpha \in \mathcal{N}_N\}$ dfined by the generating function

$$F_k(x, y) := \sum_\alpha p_\alpha(x) y^\alpha = \prod_{i=1}^N \left\{ (1 - x_i y_i)^{-1} \prod_{j=1}^N (1 - x_i y_j)^{-k} \right\} \tag{2.2}$$

$(x, y \in \mathbb{R}^N)$. For $\lambda \in \mathcal{N}_N^P$, ω_λ is defined to be the scalar multiple of $E_\lambda(x; 1/k)$ such that

$$\omega_\lambda = p_\lambda + \sum \{B(\beta, \lambda) p_\beta : |\beta| = |\lambda|, \beta^+ \succ \lambda\};$$

the triangularity property of $B(\beta, \lambda)$ was shown in [D3], further $B(\beta, \lambda)$ is independent of N in the sense that $B(\beta, \lambda)$ remains constant when β and λ are changed to $(\beta_1, \ldots, \beta_N, 0, 0, \ldots, 0)$, $(\lambda_1, \ldots, \lambda_N, 0, \ldots, 0) \in \mathcal{N}_M$ respectively (and $M \geq N$). Note that the triangularity for $\{p_\alpha\}$ is in the opposite direction to that of $\{x^\alpha\}$. Next we define the linear space E_λ as the span of the S_N-orbit of ω_λ, with basis $\{\omega_\alpha : \alpha^+ = \lambda\}$ where $\omega_{w\lambda} := w\omega_\lambda$ for $w \in S_N$ (this is well defined, since $w\lambda = \lambda$ implies $w\omega_\lambda = \omega_\lambda$ for $w \in S_N$).

The intertwining operator of type A is the unique linear map V on polynomials which satisfies: $V1 = 1$, $V : P_n \to P_n$ for each $n = 0, 1, 2, \ldots$, and $V\left(\frac{\partial}{\partial x_i} p\right)(x) = T_i(Vp)(x)$ for $1 \leq i \leq N$, $x \in \mathbb{R}^N$, each polynomial p (see [D3]). Let ξ be the linear map on polynomials defined by $\xi : p_\alpha \mapsto x^\alpha / \alpha!$, $\alpha \in \mathcal{N}_N$ and extended by linearity; then each E_λ is an eigenmanifold for $V\xi$ and $V\xi\omega_\alpha = ((Nk+1)_{\alpha^+})^{-1} \omega_\alpha$, each α.

We will use $\langle f, g\rangle$ to denote inner products (of polynomials f, g) which satisfy two conditions: $T_i x_i$ is self-adjoint for each i, and the inner product is S_N-invariant; that is, $\langle T_i x_i f(x), g(x)\rangle = \langle f(x), T_i x_i g(x)\rangle$ and $\langle wf, g\rangle = \langle f, w^{-1}g\rangle$, $w \in S_N$. The irreducibility properties of E_λ imply that such inner products are uniquely determined up to a constant on each E_λ.

Definition 2.1 *For polynomials $f(x) = \sum_\alpha f_\alpha x^\alpha$, $g(x) = \sum_\alpha g_\alpha x^\alpha$, define the A-inner product*

$$\langle f, g\rangle_A := \sum_{\alpha,\beta} f_\alpha g_\beta T^\alpha x^\beta \Big|_{x=0},$$

and the p-inner product

$$\langle f, g\rangle_p := \sum_{\alpha,\beta} f_\alpha g_\beta (H^{-1})_{\alpha\beta},$$

where the matrix H is defined by

$$F_k(x, y) = \sum_{\alpha,\beta} H_{\alpha\beta} x^\alpha y^\beta.$$

Alternatively, $\langle x^\alpha, p_\beta(x)\rangle_p = \delta_{\alpha\beta}$.

Homogeneous polynomials of different degrees are orthogonal in both inner products. The A-product was introduced in [D2] and shown to be positive-definite.

Proposition 2.2 *The operators $T_i x_i$ are self-adjoint in the p- and A-inner products.*

Proof. The adjoint of multiplication by x_i in the A-product is clearly T_i. For the p-product, self-adjointness is equivalent to

$$T_i^{(x)} x_i F_k(x, y) = T_i^{(y)} y_i F_k(x, y)$$

(the superscripts refer to the variables being acted on); but

$$T_i^{(x)} x_i F_k(x, y) = F_k(x, y) \left\{ 1 + \frac{(k+1)x_i y_i}{1 - x_i y_i} + k \sum_{j \neq i} \frac{1 - x_j y_j}{(1 - x_i y_j)(1 - x_j y_i)} \right\},$$

which is symmetric under the interchange of x and y. □

Proposition 2.3 *For polynomials f, g and $w \in S_N$, $\langle f, g\rangle_p = \langle wf, wg\rangle_p$ and $\langle f, g\rangle_A = \langle wf, wg\rangle_A$. In particular, the transpositions (ij) are self-adjoint.*

Proof. The first part follows from the equation $F_k(xw, yw) = F_k(x, y)$. The second part depends on the transformation properties of T_i, namely $w^{-1} T_i w = T_{w^{-1}(i)}$, $1 \leq i \leq N$, $w \in S_N$. □

Corollary 2.4 *For partitions λ, μ with $\lambda \neq \mu$, $E_\lambda \perp E_\mu$ in both p- and A-products.*

Proof. The method of ([D3], Theorem 4.3) used only the self-adjointness of each $T_i x_i$. $\square$

Sahi [Sa] proved this orthogonality for the p-product. In [D3] we used the modification $T_i\rho_i = T_i x_i + k$, where $\rho_i p_\alpha = p(\alpha_1, \ldots, \alpha_i + 1, \ldots)$ "raising" operator; see also [D4].

Proposition 2.5 *For $\lambda \in \mathcal{N}_N^P$, $f, g \in E_\lambda$, $\langle f, g\rangle_A = (Nk+1)_\lambda \langle f, g\rangle_p$.*

Proof. Since $g \in E_\lambda$, $\sum\{p_\alpha T^\alpha g : \ \alpha \in \mathcal{N}_N, |\alpha| = |\lambda|\} = (Nk+1)_\lambda g$ (formula for $(V\xi)^{-1}$). That is, $T^\alpha g$ is $(Nk+1)_\lambda$ times the coefficient of p_α in the expansion of g in the basis $\{p_\beta\}$. Let $f = \sum_\alpha f_\alpha x^\alpha$, $g = \sum_\beta g_\beta p_\beta$, then

$$\begin{aligned} \langle f, g\rangle_A &= \sum_\alpha f_\alpha T^\alpha g = \sum_\alpha f_\alpha g_\alpha (Nk+1)_\lambda \\ &= (Nk+1)_\lambda \langle f, g\rangle_p. \qquad \square \end{aligned}$$

Corollary 2.6 *Let $f \in E_\lambda$, then $f(T)^*1 = (Nk+1)_\lambda f$ where $f(T)^*$ denotes the p-adjoint of the operator.*

Proof. For any $\mu \in \mathcal{N}_N^P$, $g \in E_\mu$,

$$\begin{aligned} \langle g, f(T)^*1\rangle_p &= \langle f(T)g, 1\rangle_p = \langle f, g\rangle_A \\ &= \delta_{\mu\lambda}(Nk+1)_\lambda \langle f, g\rangle_p. \end{aligned}$$

Since g and μ are arbitrary, $f(T)^*1 = (Nk+1)_\lambda f$. $\square$

In [D3] we showed that

$$\begin{aligned} T_i\rho_i\omega_\alpha &= (Nk - k\#\{s : \ \alpha_s > \alpha_i\} + \alpha_i + 1)\omega_\alpha \\ &\quad + k\sum\{(ij)\omega_\alpha : \ \alpha_j > \alpha_i\}, \text{ for } \alpha \in \mathcal{N}_N, \ 1 \le i \le N. \end{aligned}$$

Also the commutator $[T_i\rho_i, T_j\rho_j] = k(T_i\rho_i - T_j\rho_j)(ij)$, for $i \neq j$ ([D3], Lemma 2.5(iii)).

This leads to the pairwise commuting operators $U_i := T_i\rho_i - k\sum_{j<i}(ij)$. (These were used by Lapointe and Vinet [[LV1], [LV2]], see also Cherednik [C].)

Definition 2.7 *For $\alpha \in \mathcal{N}_N$, $1 \le i \le N$, let*

$$\kappa_i(\alpha) := Nk - k(\#\{s : \ \alpha_s > \alpha_i\} + \#\{s : \ s < i \text{ and } \alpha_s = \alpha_i\}) + \alpha_i + 1.$$

These will appear as eigenvalues of U_i, because

$$\begin{aligned} U_i\omega_\alpha &= \kappa_i(\alpha)\omega_\alpha + k\Sigma\{(ij)\omega_\alpha : \ i < j \text{ and } \alpha_i < \alpha_j\} \\ &\quad - k\Sigma\{(ij)\omega_\alpha : \ j < 1 \text{ and } \alpha_j < \alpha_i\}. \end{aligned}$$

For each partition λ, the matrix of U_i with respect to the basis $\{\omega_\alpha : \ \alpha^+ = \lambda\}$ for E_λ is triangular in the dominance ordering (also for the lexicographic order, a total one). Note if $\alpha \in \mathcal{N}_N$, and $\alpha_i < \alpha_j, \ i < j$ for some i, j, then $(ij)\alpha \succ \alpha$.

The operators U_i satisfy some commutation properties with adjacent transposition:

$$\begin{aligned} &\text{(i)} \quad [U_i, (j, j+1)] = 0, \ \text{ if } i < j \text{ or } j+1 < i; \\ &\text{(ii)} \quad (i, i+1)U_i(i, i+1) = U_{i+1} + k(i, i+1). \end{aligned} \tag{2.3}$$

Theorem 2.8 *([D4], Section 3) For each partition λ there exists a unique basis $\{\zeta_\alpha : \ \alpha^+ = \lambda\}$ for E_λ satisfying*

(1) $U_i\zeta_\alpha = \kappa_i(\alpha)\zeta_\alpha, \ 1 \leq i \leq N;$

(2) $\zeta_\alpha = \omega_\alpha + \Sigma\{B(\beta, \alpha)\omega_\beta : \ \beta^+ = \lambda$ *and* $\beta \succ \alpha\}$.

(3) $\langle \zeta_\alpha, \zeta_\beta \rangle = 0$ *if* $\alpha \neq \beta$.

Corollary 2.9 $\zeta_\lambda = \omega_\lambda$, *and if* $\alpha_i = \alpha_{i+1}$, *then* $(i, i+1)\zeta_\alpha = \zeta_\alpha$.

Proof. The partition λ is the maximum element in $\{\alpha : \ \alpha^+ = \lambda\}$.

Suppose $\alpha_i = \alpha_{i+1}$, and expand $(i, i+1)\zeta_\alpha$ in the basis $\{\zeta_\beta : \ \beta^+ = \lambda = \alpha^+\}$. Because $(i, i+1)\zeta_\alpha$ is an eigenvector for each U_j with $|i - j| > 1$ with eigenvalue $\kappa_i(\alpha)$, it must be a scalar multiple of ζ_α. The fact that $(i, i+1)\omega_\alpha = \omega_\alpha$ and (2.3)(ii) shows the factor is 1. $\square$

Proposition 2.10 *Suppose $\alpha \in \mathcal{N}_N$ and $\alpha_i > \alpha_{i+1}$, then span$\{\zeta_\alpha, \zeta_{\sigma\alpha}\}$ is invariant under $\sigma = (i, i+1)$, and the matrix of σ in this basis is*

$$\begin{bmatrix} c & 1 - c^2 \\ 1 & -c \end{bmatrix} \ \textit{where } c = \frac{k}{\kappa_i(\alpha) - \kappa_{i+1}(\alpha)}.$$

Proof. Let $g = \sigma\zeta_\alpha - c\zeta_\alpha$, then $U_j g = \kappa_j(\alpha)g$ for $j < i$ or $i + 1 < j$; this shows $g \in$ span$\{\zeta_\alpha, \zeta_{\sigma\alpha}\}$. The coefficient of $\omega_{\sigma\alpha}$ in g is 1, since $\alpha \succ \sigma\alpha$. The commutation relation $\sigma U_i \sigma = U_{i+1} + k\sigma$ shows $U_i g = \kappa_{i+1}(\alpha)g$ and $U_{i+1}g = \kappa_i(\alpha)g$, but $\kappa_{i+1}(\sigma\alpha) = \kappa_i(\alpha)$ and $\kappa_{i+1}(\alpha) = \kappa_i(\sigma\alpha)$, thus $g = \zeta_{\sigma\alpha}$. Finally, $\sigma g = \zeta_\alpha - c\sigma\zeta_\alpha = (1 - c^2)\zeta_\alpha - cg$. $\square$

These equations were found by Sahi [Sa], see also Baker and Forrester [BF3].

In the basis $\{E_\alpha(x; 1/k), \, E_{\sigma\alpha}(x; 1/k)\}$ the matrix of σ is $\left[\begin{smallmatrix} c & 1 \\ 1-c^2 & -c \end{smallmatrix}\right]$. By use of the known evaluations at $x = 1^N$ and the notation of Definitions 3.10 and 3.17,

$$E_\alpha(x; 1/k) = \frac{h(\alpha^+, 1)}{h(\alpha^+, k+1)\mathcal{E}_+(\alpha)\mathcal{E}_-(\alpha)} \, \zeta_\alpha(x).$$

Corollary 2.11 *Suppose $\alpha \in \mathcal{N}_N$ and $\alpha_i > \alpha_{i+1}$, then*

$$\zeta_{\sigma\alpha}(1^N) = (1-c)\zeta_\alpha(1^N),$$

and

$$\|\zeta_{\sigma\alpha}\|^2 = (1-c^2)\|\zeta_\alpha\|^2,$$

for $c = \frac{k}{\kappa_i(\alpha)-\kappa_{i+1}(\alpha)}$ and $\sigma = (i, i+1)$.

Proof. Since $\zeta_{\sigma\alpha} = \sigma\zeta_\alpha - c\zeta_\alpha$, we have that

$$\begin{aligned} \zeta_{\sigma\alpha}(1^N) &= \sigma\zeta_\alpha(1^N) - c\zeta_\alpha(1^N) \\ &= (1-c)\zeta_\alpha(1^N). \end{aligned}$$

Since σ is self-adjoint the matrix of σ in the orthonormal basis $\{\zeta_\alpha/\|\zeta_\alpha\|, \zeta_{\sigma\alpha}/\|\zeta_{\sigma\alpha}\|\}$ must be symmetric, hence $\|\zeta_{\sigma\alpha}\|^2 = (1-c^2)\|\zeta_\alpha\|^2$. □

Corollary 2.12 *In the same notation, let*

$$f_0 = \zeta_\alpha + \left(\frac{1}{1+c}\right)\zeta_{\sigma\alpha}, \ \textit{and}\ f_1 = \zeta_\alpha - \left(\frac{1}{1-c}\right)\zeta_{\sigma\alpha}, \tag{2.4}$$

then $\sigma f_0 = f_0$ and $\sigma f_1 = -f_1$.

In the next section we derive expressions for the norms $\|\zeta_\alpha\|_p^2$, $\|\zeta_\alpha\|_A^2$ as a by-product. If $\alpha \in \mathcal{N}_N$ and $\alpha_i > \alpha_{i+1}$, then $\kappa_i(\alpha) - \kappa_{i+1}(\alpha) \geq \alpha_i - \alpha_{i+1} + k$, thus $0 < c < 1$ (when $k > 0$); in fact,

$$\begin{aligned} \kappa_i(\alpha) - \kappa_{i+1}(\alpha) &= \alpha_i - \alpha_{i+1} + k(1 + \#\{s:\ s > i \text{ and } \alpha_s = \alpha_i\} \\ &\quad + \#\{s:\ s < i \text{ and } \alpha_s = \alpha_{i+1}\} + \#\{s:\ \alpha_{i+1} < \alpha_s < \alpha_i\}). \end{aligned}$$

The relation of the operator U_i to the Hamiltonian $\mathcal{H}_1$ in (1.2) is as follows: let

$$h(x) = \prod_{1\leq i<j\leq N} |x_i - x_j|^k \prod_{i=1}^N |x_i|^{k(N-1)/2}$$

(an S_N-invariant positively homogeneous function), then

$$\begin{aligned} &h(x)(U_i - 1 - k(N+1)/2)(f(x)/h(x)) \\ &:= \quad A_i f(x) = x_i \frac{\partial f(x)}{\partial x_i} - k \sum_{j\neq i} \frac{x_{\max(i,j)}}{x_i - x_j}\,(ij) f(x), \end{aligned}$$

and $\sum_{i=1}^N A_i^2 = \mathcal{H}_1$. The eigenvalue of $\mathcal{H}_1$ on the space $h(x)E_\lambda$ is

$$\sum_{i=1}^N (\kappa_i(\lambda) - 1 - k(N+1)/2)^2 = \sum_{i=1}^N \lambda_i^2 + k\sum_{i=1}^N (N-2i+1)\lambda_i + k^2 N(N^2-1)/12$$

(see Baker and Forrester [BF1], [BF2], Lapointe and Vinet [LV2]); the last term is the energy of the ground state. In the coordinates $x_j = \exp(\sqrt{-1}\,\theta_j)$,

$$h = 2^k \prod_{i<j} |\sin((\theta_i - \theta_j)/2)|^k.$$

3 Subgroup Invariants

A parabolic subgroup of S_N is by definition generated by a subset of $\{(i, i+1) : 1 \leq i < N\}$. This section concerns subspaces of E_λ invariant under a parabolic subgroup. We start with the basic structure, an interval and its group of permutations. A typical interval is denoted by I or $[\ell_1, \ell_2]$ and is defined to be $\{n \in \mathbb{Z} : \ell_1 \leq n \leq \ell_2\}$. The associated permutation group, denoted by S_I, is defined as $\{w \in S_N : w(i) = i \text{ for all } i \notin I\}$. Thus $S_I \cong S_m$ where $m = \#I$. The parabolic subgroups of S_N are direct products of such groups corresponding to a collection of disjoint intervals in $[1, N]$.

The technique developed in this section will be used to derive formulas for the norms of the non-symmetric Jack polynomials ζ_α, as well as for polynomials with prescribed symmetric or skew-symmetric properties for parabolic subgroups.

For any $\beta \in \mathcal{N}_N$, the set $\{w\beta : w \in S_I\}$ has a unique $\succ$-maximal element, which satisfies the following:

Definition 3.1 *For an interval I, say that a composition α satisfies property $(\geq, I)$ or $(>, I)$ if $\alpha_i \geq \alpha_j$, respectively $\alpha_i > \alpha_j$, whenever $i, j \in I$ and $i < j$.*

We deal with the case of one interval first. Let $I = [\ell+1, \ell+m]$ with $1 \leq \ell+1 < \ell+m \leq N$. The object is to analyze $\operatorname{span}\{\zeta_{w\alpha} : w \in S_I\}$ and $\operatorname{span}\{w\zeta_\alpha : w \in S_I\}$ for a fixed α satisfying $(\geq, I)$. The structure of $\operatorname{span}\{\zeta_{w\alpha}\}$ mimics that of E_λ (with m variables) with an analogue of $T_i\rho_i$. Part of the motivation for the following definition is to have commutativity among the operators associated with disjoint intervals.

Definition 3.2 *For a fixed interval $I = [\ell+1, \ell+m]$, for $i \in I$, let*

$$\tau_i := T_i\rho_i - k\sum_{j\leq\ell}(ij).$$

Note that $U_i = \tau_i - k\sum_{\ell<j<i}(ij)$, and $w^{-1}\tau_{w(i)}w = \tau_i$ for $w \in S_I$.

Now fix a composition α satisfying $(\geq, I)$ and let $X = \operatorname{span}\{\zeta_{w\alpha} : w \in S_I\}$, a linear space with $\dim X = \#(S_I\alpha)$, a subspace of E_{α^+}. Then X is invariant under S_I because S_I is generated by $\{(i, i+1) : \ell+1 \leq i < \ell+m\}$ and Proposition 2.9 applies.

Definition 3.3 *For $w \in S_I$ let $g_{w\alpha} := w\zeta_\alpha$. This is well-defined because $w_1\alpha = w_2\alpha$ implies $(w_2^{-1}w_1)\zeta_\alpha = \zeta_\alpha$; since the subgroup $\{w \in S_I : w\alpha = \alpha\}$ is generated by adjacent transpositions (from the condition $(\geq, I)$), Corollary 2.10.*

Note that it is not generally true that $w\beta = \beta$ for some composition β and $w \in S_N$ implies $w\zeta_\beta = \zeta_\beta$, for example (1,3) $\zeta_{(1,2,1)} \neq \zeta_{(1,2,1)}$.

Proposition 3.4 $X = span\{g_\beta : \ \beta \in S_I\alpha\}$.

Proof. Consider the set $A = \{\gamma : \ \zeta_\gamma \in \text{span}\{g_\beta\}\}$, by Proposition 2.10, if $\gamma \in A$ and $\gamma_i \neq \gamma_{i+1}$ for $\ell + 1 \leq i < \ell + m$, then $(i, i+1)\gamma \in A$. Also $\alpha \in A$, hence $A = S_I\alpha$. □

Proposition 3.5 *For $\beta \in S_I\alpha$, and $i \in I$,*

$$\tau_i g_\beta = \kappa_i'(\beta) g_\beta + k \sum \{(ij) g_\beta : \ j \in I \ \textit{and} \ \beta_j > \beta_i\},$$

where

$$\kappa_i'(\beta) = Nk - k(\#\{s : \ \beta_s > \beta_i\} + \#\{s : \ s \leq \ell \ \textit{and} \ \beta_s = \beta_i\} + \beta_i + 1.$$

Proof. First for $\beta = \alpha$,

$$\begin{aligned} \tau_i g_\alpha &= \left(U_i + k \sum_{\ell < j < i} (ij) \right) g_\alpha \\ &= (\kappa_i(\alpha) + k\#\{s : \ \ell < s < i \text{ and } \alpha_s = \alpha_i\}) g_\alpha \\ &\quad + k \sum \{(ij) g_\alpha : \ \ell < j < i \text{ and } \alpha_j > \alpha_i\}, \end{aligned} \tag{3.1}$$

because $g_\alpha = \zeta_\alpha$. This is the required formula for this case; the situation $\{j : \ i < j \leq \ell + m \text{ and } \alpha_j > \alpha_i\}$ does not occur.

For an arbitrary $w \in S_I$, let $s = w^{-1}(i)$, $\beta = w\alpha$, then

$$\begin{aligned} \tau_i g_\beta &= \tau_i w g_\alpha = w \tau_s g_\alpha \\ &= (Nk - k(\#\{j : \ \alpha_j > \alpha_s\} + \#\{j : \ j \leq \ell, \, \alpha_j = \alpha_s\} + \alpha_s + 1) w g_\alpha \\ &\quad + k \sum \{w(s,j) g_\alpha : \ \ell < j \leq \ell + m, \ \alpha_j > \alpha_s\}, \end{aligned}$$

but $w(s,j) = (w(s), w(j))w = (i, w(j))w$, $\beta_t = \alpha_{w^{-1}(t)}$ for any t, $\beta_i = \alpha_s$. Thus

$$\tau_i g_\beta = \kappa_i'(\beta) g_\beta + k \sum \{(i, w(j)) g_\beta : \ j \in I \text{ and } \alpha_j > \beta_i\},$$

and $\alpha_j = \beta_{w(j)}$. □

This showed that the structure of the operators $\{\tau_i : \ i \in I\}$ on X is essentially the same as that of $\{T_i \rho_i : \ 1 \leq i \leq N\}$ on E_λ.

Proposition 3.6 *Suppose C is a linear operator on X and $[C, \tau_i] = 0$ for each $i \in I$, then $C = c1$, a multiple of the identity.*

Proof. The same proof as ([D3], Proposition 3.2) for E_λ works, replacing $\{\omega_\beta : \beta^+ = \lambda\}$ by $\{g_\beta : \beta \in S_I\alpha\}$. $\square$

Proposition 3.7 *The operator $U_I := \prod_{i\in I} U_i$ commutes with each $w \in S_I$; also $[U_I, \tau_i] = 0$ for $i \in I$ and $U_I g = \prod_{i\in I} \kappa_i(\alpha) g$ for each $g \in X$.*

Proof. To show $[U_I, w] = 0$ for $w \in S_I$ it suffices to prove this for $w = (i, i+1)$, $\ell + 1 \le i < \ell + m$. Also $wU_j = U_j w$ if $j < i$ or $j > i + 1$, thus consider

$$\begin{aligned} wU_iU_{i+1}w &= (U_{i+1}w + k)U_{i+1}w \\ &= U_{i+1}(U_iw - k)w + kU_{i+1}w \\ &= U_{i+1}U_i \\ &= U_iU_{i+1} \end{aligned}$$

(by (2.3)). Since $\tau_{\ell+1} = U_{\ell+1}$ we have $[U_I, \tau_{\ell+1}] = 0$.

For any $w \in S_I$, $\tau_{\ell+1} = w^{-1}\tau_{w(\ell+1)}w$, and this shows $[U_I, \tau_j] = 0$ for each $j \in I$.

For any basis element $g_{w\alpha}$ of X, $U_I g_{w\alpha} = wU_I g_\alpha = w \prod_{i\in I} \kappa_i(\alpha) g_\alpha$ since $g_\alpha = \zeta_\alpha$. $\square$

The change of basis matrix for $\{g_\beta\}$ to $\{\zeta_\beta\}$ is triangular; define B by

$$\zeta_\beta = \sum_{\gamma\in S_I\alpha} B(\gamma, \beta) g_\gamma, \quad \text{then } B(\gamma, \gamma) = 1, \tag{3.2}$$

and $B(\gamma, \beta) = 0$ unless $\gamma \succeq \beta$. The usual proof (for $\{\omega_\beta\}$ and $\{\zeta_\beta\}$) applies. There is a nice relationship between B and the Gram matrix for $\{g_\beta\}$.

Definition 3.8 *For $\beta, \gamma \in S_I\alpha$, let*

$$H(\beta, \gamma) := \langle g_\beta, g_\gamma\rangle / \|g_\alpha\|^2$$

(independent of choice of permissible inner product).

Proposition 3.9 *For $w_1, w_2 \in S_I$, and $\beta, \gamma \in S_I\alpha$,*

$$H(w_1\beta, w_2\gamma) = H(\beta, w_1^{-1}w_2\gamma);$$

in particular,

$$H(w_1\alpha, w_2\alpha) = H(\alpha, w_1^{-1}w_2\alpha) = B^{-1}(\alpha, w_1^{-1}w_2\alpha).$$

Proof. The first identity follows from the S_N-invariance of the inner product. For $\beta \in S_I\alpha$,

$$H(\alpha,\beta) = \langle \zeta_\alpha, \sum_{\gamma \succeq \beta} B^{-1}(\gamma,\beta)\zeta_\gamma\rangle / \|\zeta_\alpha\|^2 = B^{-1}(\alpha,\beta)$$

(by orthogonality, $g_\alpha = \zeta_\alpha$). □

Sahi [Sa] found a formula for $\|\zeta_\beta\|_p^2$ in terms of a hook length product associated to the Ferrers diagram of the composition β. Here we give an expression whose complexity is roughly the number of adjacent transpositions needed to transform α to β; the upper and lower hook length products for partitions (Stanley [St]) will also be used eventually.

Definition 3.10 *For $\epsilon = +$ or $-$ ("sign"), an interval I, $\beta \in \mathcal{N}_N$, let*

$$\mathcal{E}_\epsilon(\beta, I) := \prod \left\{ 1 + \frac{\epsilon k}{\kappa_j(\beta) - \kappa_i(\beta)} : \ \beta_i < \beta_j, \ i < j, \ \text{and } i, j \in I \right\}.$$

Observe $\mathcal{E}_\epsilon(\alpha, I) = 1$ (α satisfies $(\geq, I)$).

Lemma 3.11 *If $\beta_{i+1} > \beta_i$ for $\ell + 1 \leq i < \ell + m$, then*

$$\mathcal{E}_\epsilon((i,i+1)\beta, I)/\mathcal{E}_\epsilon(\beta, I) = 1 + \frac{\epsilon k}{\kappa_i(\beta) - \kappa_{i+1}(\beta)}, \ \epsilon = \pm.$$

Proof. For $\{i,j\} \subset I$ let $t(\beta; i, j) = 1 + \frac{\epsilon k}{\kappa_j(\beta) - \kappa_i(\beta)}$ if $\beta_i < \beta_j$ and $i < j$, else $t(\beta; i, j) = 1$. Recall

$$\kappa_j(\beta) = Nk - k(\#\{s : \ \beta_s > \beta_j\} + \#\{s : \ s < j, \ \beta_s = \beta_j\}) + \beta_j + 1,$$

each j.

Let $\sigma = (i, i+1)$. Then $t(\sigma\beta; i, j) = t(\beta; i+1, j)$ and $t(\sigma\beta; i+1, j) = t(\beta; i, j)$ for $j > i+1$; $t(\sigma\beta; j, i) = t(\beta; j, i+1)$ and $t(\sigma\beta; j, i+1) = t(\beta; j, i)$ for $j < i$. Also $t(\beta; i, i+1) = 1$, and

$$t(\sigma\beta; i, i+1) = 1 + \frac{\epsilon k}{\kappa_{i+1}(\sigma\beta) - \kappa_i(\sigma\beta)} = 1 + \frac{\epsilon k}{\kappa_i(\beta) - \kappa_{i+1}(\beta)}.$$

The values $t(\beta; j_1, j_2) = t(\sigma\beta; j_1, j_2)$ for indices (j_1, j_2) not listed above. Since $\mathcal{E}_\epsilon(\beta; I) = \prod_{i,j \in I} t(\beta; i, j)$ this shows $\mathcal{E}_\epsilon(\sigma\beta; I)/\mathcal{E}_\epsilon(\beta; I)$ has the specified value. □

Proposition 3.12 *Suppose $\beta \in S_I\alpha$, then*

$$\zeta_\beta(1^N) = \mathcal{E}_-(\beta; I)\zeta_\alpha(1^N),$$

and

$$\|\zeta_\beta\|^2 = \mathcal{E}_+(\beta; I)\mathcal{E}_-(\beta; I)\|\zeta_\alpha\|^2.$$

Proof. Corollary 2.11 showed that

$$\frac{\|\zeta_{\sigma\beta}\|^2}{\|\zeta_\beta\|^2} = \frac{\mathcal{E}_+(\sigma\beta; I)\mathcal{E}_-(\sigma\beta; I)}{\mathcal{E}_+(\beta; I)\mathcal{E}_-(\beta; I)}$$

for $\beta_i > \beta_{i+1},\ \ell+1 \le i < \ell+m$, and $\sigma := (i, i+1)$. The transpositions $(i, i+1)$ generate S_I. Similarly, $\zeta_\beta(1^N)/\mathcal{E}_-(\beta; I)$ is constant on $S_I\alpha$. $\square$

Observe that the case $I = [1, N]$ provides the values $\|\zeta_\beta\|^2/\|\zeta_\lambda\|^2$ and $\zeta_\beta(1^N)/\zeta_\lambda(1^N)$ for $\lambda = \beta^+$.

The minimum (for $\succeq$) element in $S_I\alpha$ occurs in several important formulas. It is denoted by

$$\alpha^R := (\alpha_1, \ldots, \alpha_\ell, \alpha_{\ell+m}, \alpha_{\ell+m-1}, \ldots, \alpha_{\ell+1}, \alpha_{\ell+m+1}, \ldots, \alpha_N),$$

that is, $\alpha^R = \sigma_I\alpha$ where σ_I is the longest element in S_I (the length of a permutation is the minimum number of adjacent transpositions needed to produce it, as a product). Thus $\sigma_I = (\ell+m, \ell+1)(\ell+m-1, \ell+2)\ldots$ and is an involution, $\sigma_I^2 = 1$.

Lemma 3.13

$$\mathcal{E}_\epsilon(\alpha^R; I) = \prod \left\{1 + \frac{\epsilon k}{\kappa_i(\alpha) - \kappa_j(\alpha)} :\ \ell+1 \le i < j \le \ell+m \text{ and } \alpha_i > \alpha_j\right\}.$$

Proof. When $\alpha_i = \alpha_{i+1}$ for some $i \in I$, then the list of eigenvalues $\kappa_{\ell+1}(\alpha^R), \ldots, \kappa_{\ell+m}(\alpha^R)$ is not the reverse of the list $\kappa_{\ell+1}(\alpha), \ldots, \kappa_{\ell+m}(\alpha)$, nevertheless the relative order of pairwise different values is reversed; that is, $\alpha_i^R > \alpha_j^R$ if and only if $\alpha_{2\ell+m+1-i} > \alpha_{2\ell+m+1-j}$ (for $\ell+1 \le j < i \le \ell+m$). This suffices to establish the formula. $\square$

As an example for the case $\alpha_i = \alpha_{i+1}$, take $\ell = 0,\ m = 3$, and $\alpha = (\alpha_1, \alpha_1, \alpha_3)$ with $\alpha_1 > \alpha_3$; then $\kappa_1(\alpha^R) = \kappa_3(\alpha),\ \kappa_2(\alpha^R) = \kappa_1(\alpha),\ \kappa_3(\alpha^R) = \kappa_2(\alpha)$.

There is a unique S_I-invariant in X, and if α satisfies $(>, I)$ there is a unique S_I-alternating polynomial in X. Invariance for S_I means $wf = f$, "alternating" means $wf = \operatorname{sgn}(w)f$, for all $w \in S_I$. To establish such properties it suffices to show $(i, i+1)f = f$ for invariance, $(i, i+1)f = -f$ for alternating, for $\ell+1 \le i < \ell+m$.

Definition 3.14 *Let*

$$j_{\alpha;I} := \mathcal{E}_+(\alpha^R; I) \sum_{\beta \in S_I\alpha} \frac{1}{\mathcal{E}_+(\beta; I)}\ \zeta_\beta;$$

and

$$a_{\alpha;I} := \mathcal{E}_-(\alpha^R; I) \sum_{w \in S_I} \frac{sgn(w)}{\mathcal{E}_-(w\alpha; I)}\ \zeta_{w\alpha},$$

provided α satisfies $(>, I)$.

Theorem 3.15 *The polynomials $j_{\alpha;I}$ and $a_{\alpha;I}$ have the following properties:*

(1) $wj_{\alpha;I} = j_{\alpha;I}$ *for* $w \in S_I$*;*

(2) $\|j_{\alpha;I}\|^2 = (\#S_I\alpha)\mathcal{E}_+(\alpha^R;I)\|\zeta_\alpha\|^2$*;*

(3) $j_{\alpha;I}(1^N) = (\#S_I\alpha)\zeta_\alpha(1^N)$*;*

(4) $j_{\alpha;I} = \sum_{\beta\in S_I\alpha}\ g_\beta$*;*

(5) $wa_{\alpha;I} = (sgn\ w)a_{\alpha;I}$ *for* $w \in S_I$*;*

(6) $\|a_{\alpha;I}\|^2 = (\#S_I)\mathcal{E}_-(\alpha^R;I)\|\zeta_\alpha\|^2$*;*

(7) $a_{\alpha;I} = \sum_{w\in S_I}(sgn\ w)g_{w\alpha}$*.*

Proof. It is clear that the polynomials defined in (4) and (7) are the only (up to scalar multiple) invariant and alternating elements of X. To show (1) and (5), consider a typical element $f = \sum_{\beta\in S_I\alpha} f_\beta\zeta_\beta$, and fix i with $\ell+1 \le i < \ell+m$, and let $\sigma = (i,i+1)$. Then

$$\begin{aligned} f &= \sum\{f_\beta\zeta_\beta :\ \beta_i = \beta_{i+1},\ \beta\in S_I\alpha\} \\ &\quad + \sum\{f_\beta\zeta_\beta + f_{\sigma\beta}\zeta_{\sigma\beta} :\ \beta_i > \beta_{i+1},\ \beta\in S_I\alpha\}. \end{aligned}$$

By Corollary 2.12, $\sigma f = f$ if and only if

$$\frac{f_{\sigma\beta}}{f_\beta} = \frac{\mathcal{E}_+(\beta;I)}{\mathcal{E}_+(\sigma\beta;I)} = \left(1 + \frac{k}{\kappa_i(\beta)-\kappa_{i+1}(\beta)}\right)^{-1},$$

for each $\beta \in S_I\alpha$ with $\beta_i > \beta_{i+1}$; also $\sigma f = -f$ if and only if $\beta_i = \beta_{i+1}$ implies $f_\beta = 0$ and $\beta_i > \beta_{i+1}$ implies

$$\frac{f_{\sigma\beta}}{f_\beta} = -\frac{\mathcal{E}_-(\beta;I)}{\mathcal{E}_-(\sigma\beta;I)}.$$

Note if α does not satisfy $(>,I)$ then there is no nonzero alternating element. By the triangularity of B, the coefficient of g_{α^R} in $j_{\alpha,I}$ or $a_{\alpha;I}$ is the same as the coefficient of ζ_{α^R}; this establishes (4) and (7).

To compute $\|j_{\alpha,I}\|^2$, observe that $\langle g_\beta, \sum_{\gamma\in S_I\alpha} g_\gamma\rangle = \langle g_\alpha, \sum_\gamma g_\gamma\rangle$ for each $\beta\in S_I\alpha$ by the S_N-invariance of the inner product. Sum this equation over all $\beta\in S_I\alpha$ to obtain

$$\langle j_{\alpha;I}, j_{\alpha;I}\rangle = (\#S_I\alpha)\langle\zeta_\alpha, j_{\alpha;I}\rangle = (\#S_I\alpha)\mathcal{E}_+(\alpha^R;I)\|\zeta_\alpha\|^2$$

(using the original definition of $j_{\alpha;I}$).

Formula (3) is a direct consequence of (4). The calculation for $\|a_{\alpha;I}\|^2$ proceeds similarly:

$$\|a_{\alpha;I}\|^2 = \sum_{w_1\in S_I}\sum_{w_2\in S_I} \mathrm{sgn}(w_1)\mathrm{sgn}(w_2)\langle g_{w_1\alpha}, g_{w_2\alpha}\rangle$$

$$
\begin{aligned}
&= (\#S_I)\sum_{w_2\in S_I}\operatorname{sgn}(w_2)\langle g_\alpha, g_{w_2\alpha}\rangle \\
&= (\#S_I)\langle \zeta_\alpha, a_{\alpha;I}\rangle \\
&= (\#S_I)\mathcal{E}_-(\alpha^R;I)\|\zeta_\alpha\|^2
\end{aligned}
$$

(replacing w_2 by w_1w_2 in the inner sum.) □

Corollary 3.16

$$\sum_{\beta\in S_I\alpha} H(\alpha,\beta) = \mathcal{E}_+(\alpha^R;I),$$

and if α satisfies $(>,I)$, then

$$\sum_{w\in S_I} sgn(w)H(\alpha,w\alpha) = \mathcal{E}_-(\alpha^R;I).$$

Proof. By definition of H,

$$\sum_{\beta\in S_I\alpha} H(\alpha,\beta) = \langle g_\alpha, \sum_\beta g_\beta\rangle/\|g_\alpha\|^2 = \langle g_\alpha, j_{\alpha;I}\rangle/\|g_\alpha\|^2,$$

and

$$\sum_{w\in S_I}\operatorname{sgn}(w)H(\alpha,w\alpha) = \langle g_\alpha, a_{\alpha;I}\rangle/\|g_\alpha\|^2. \qquad \square$$

The triangularity argument for extracting the coefficient of ζ_{α^R} was used already by Baker, Dunkl, and Forrester [BDF] in the same context. Earlier, Baker and Forrester [BF2] considered some special cases of subgroup invariance and relations to the Jack polynomials.

An analogue of evaluation at 1^N for $a_{\alpha;I}$ will be discussed later. Hook length products are used in the norm calculations.

Definition 3.17 *For a partition λ and parameter t, let*

$$h(\lambda,t) := \prod_{i=1}^{N-m_0}\prod_{j=1}^{\lambda_i}(\lambda_i - j + t + k\#\{s:\ s>i \text{ and } j\le\lambda_s\le\lambda_i\}),$$

where $\lambda_i = 0$ for $i > N-m_0$. The special cases $t=1$, k satisfy $k^{-|\lambda|}h(\lambda,1) = h^*(\lambda)$, $k^{-|\lambda|}h(\lambda,k) = h_*(\lambda)$, the upper and lower hook length products for parameter $1/k$ (Stanley [St]), respectively.

In the next paragraphs, let $I=[1,N]$, and let λ be a partition and $\lambda^R=(\lambda_N,\lambda_{N-1},\dots,\lambda_1)$. We will use known results for the Jack polynomials to determine $\|\zeta_\lambda\|_p^2$; these are due to Stanley [St]. Sahi first found $\|\zeta_\lambda\|_p^2$, and recently Baker and Forrester [BF4] presented a concise self-contained determination of the structural constants of the Jack polynomials.

We apply the previous results of this section and we suppress the letter I in $j_{\lambda;I}$ and $\mathcal{E}_s(\beta;I)$ since $I=[1,N]$.

¿From the orthogonality relations on the N-torus (more on this later), it is known that $J_\lambda(x;1/k)$ is a multiple of j_λ (where J_λ is the standard Jack polynomial). Stanley showed $J_\lambda(1^N;1/k)=(Nk)_\lambda k^{-|\lambda|}$, also in ([D3], Proposition 4.3) we showed $\zeta_\lambda(1^N)=\omega_\lambda(1^N)=(Nk+1)_\lambda/h(\lambda,1)$, so for any $\beta\in\mathcal{N}_N$, by Proposition 3.12,

$$\zeta_\beta(1_N)=\mathcal{E}_-(\beta)(Nk+1)_{\beta^+}/h(\beta^+,1). \tag{3.3}$$

Thus by Theorem 3.15(3),

$$J_\lambda(x;1/k)=\frac{(Nk)_\lambda h(\lambda,1)}{(Nk+1)_\lambda k^{|\lambda|}(\#S_N\lambda)}\ j_\lambda(x).$$

Note $\#S_N\lambda=\#\{\beta:\ \beta^+=\lambda\}$, the dimension of E_λ. The ${}_1F_0$ formula for Jack polynomials (Yan [Yn], Beerends and Opdam [BO]) asserts

$$\prod_{i=1}^N(1-x_i)^{-(Nk+1)}=\sum_{\lambda\in\mathcal{N}_N^P}\frac{(Nk+1)_\lambda}{k^{|\lambda|}h_*(\lambda)h^*(\lambda)}\ J_\lambda\left(x;\frac{1}{k}\right).$$

We convert this to an expression in j_λ.

Lemma 3.18

$$h(\lambda,k)=\frac{(Nk)_\lambda\mathcal{E}_+(\lambda^R)h(\lambda,k+1)}{(Nk+1)_\lambda(\#S_N\lambda)},$$

for $\lambda\in\mathcal{N}_N^P$.

Proof. We will show

$$\frac{h(\lambda,k)(Nk+1)_\lambda}{h(\lambda,k+1)(Nk)_\lambda}=\frac{\mathcal{E}_+(\lambda^R)}{\#S_N\lambda}.$$

Denote the factors of $h(\lambda,t)$ by $h(i,j;t)=\lambda_i-j+t+k\#\{s:\ j\le\lambda_s\le\lambda_i\}$, $1\le j\le\lambda_i$. Then $h(i,j,k)=h(i,j+1,k+1)$ whenever $j\ne\lambda_s$ for any s. The ratio $h(\lambda,k)/h(\lambda,k+1)$, after cancellation, is a product of factors like $h(i,\lambda_i,k)/h(i,1,k+1)$ and $h(i,\lambda_s,k)/h(i,\lambda_s+1,k+1)$, for $\lambda_s<\lambda_i$.

We have $h(i,\lambda_i,k)=k(1+\#\{s:\ s>i,\ \lambda_s=\lambda_i\})$ and

$$\begin{aligned} h(i,1,k+1) &= \lambda_i+k+k\#\{s:\ s>i,\ 1\le\lambda_s\le\lambda_i\} \\ &= \lambda_i+k(N-m_0-i+1), \end{aligned}$$

where m_0 is the number of zero parts of λ (as element of $\mathcal{N}_N$). For each $j \in \mathbb{Z}_+$, let $m_j = \#\{i : \lambda_i = j\}$, then

$$\prod_{i=1}^{N-m_0} h(i, \lambda_i, k) = k^{N-m_0} \prod_{j \geq 1} m_j!$$

Also,

$$\begin{aligned}
\frac{(Nk+1)_\lambda}{(Nk)_\lambda} &= \prod_{i=1}^{N-m_0} \frac{\lambda_i + (N-i+1)k}{k(N-i+1)} \\
&= \frac{m_0!}{k^{N-m_0} N!} \prod_{i=1}^{N-m_0} (\lambda_i + (N-i+1)k).
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{h(\lambda, k)(Nk+1)_\lambda}{h(\lambda, k+1)(Nk)_\lambda} \\
&= \frac{m_0! \prod_{j\geq 1} m_j!}{N!} \prod_{i=1}^{N-m_0} \prod \left\{ \frac{\lambda_i - \lambda_j + k(1 + \#\{s : \ s > i, \ \lambda_j \leq \lambda_s \leq \lambda_i\})}{\lambda_i - \lambda_j + k(1 + \#\{s : \ s > i, \ \lambda_j < \lambda_s \leq \lambda_i\})} \right. \\
&\qquad \left. : \text{ distinct values of } \lambda_j < \lambda_i \right\}.
\end{aligned}$$

In the latter product $\lambda_j = 0$ is used; it contributes

$$\frac{\lambda_i + k(N-i+1)}{\lambda_i + k(N-m_0-i+1)}.$$

Since $\#S_N\lambda = N!/\prod_j m_j!$ it suffices to identify the remaining product with

$$\mathcal{E}_+(\lambda^R) = \prod \left\{ \frac{\kappa_i(\lambda) - \kappa_j(\lambda) + k}{\kappa_i(\lambda) - \kappa_j(\lambda)} : \ i < j \text{ and } \lambda_i > \lambda_j \right\}.$$

Let μ_1, μ_2 be two distinct values in $(\lambda_1, \lambda_2, \ldots, \lambda_N)$ with $\mu_1 > \mu_2$; supose $\lambda_{a_1} = \lambda_{a_1+1} = \cdots = \lambda_{a_1+n_1-1} = \mu_1$ and $\lambda_{a_2} = \cdots = \lambda_{a_2+n_2-1} = \mu_2$ (and no other appearances of μ_1, μ_2) and $a_1 + n_1 - 1 < a_2$. This implies $\kappa_t(\lambda) = (N-t+1)k + \mu_1 + 1$, $a_1 \leq t < a_1 + n_1$ and $\kappa_u(\lambda) = (N-u+1)k + \mu_2 + 1$, $a_2 \leq u < a_2 + n_2$.

The contribution of pair (μ_1, μ_2) to the left-hand product is

$$\prod_{t=a_1}^{a_1+n_1-1} \frac{\mu_1 - \mu_2 + k(a_2 + n_2 - t)}{\mu_1 - \mu_2 + k(a_2 - t)}$$

$$= \prod_{t=a_1}^{a_1+n_1-1} \prod_{u=a_2}^{a_2+n_2-1} \frac{\mu_1-\mu_2+k(u+1-t)}{\mu_1-\mu_2+k(u-t)}$$

$$= \prod_{t,u} \frac{\kappa_t(\lambda)-\kappa_u(\lambda)+k}{\kappa_t(\lambda)-\kappa_u(\lambda)},$$

which is exactly the contribution of (μ_1, μ_2) to $\mathcal{E}_+(\lambda^R)$. □

Proposition 3.19

$$\prod_{i=1}^{N}(1-x_i)^{-(Nk+1)} = \sum_{\lambda\in\mathcal{N}_N^P} \frac{(Nk+1)_\lambda}{h(\lambda,k+1)\mathcal{E}_+(\lambda^R)}\, j_\lambda(x) \quad (|x_i|<1 \textit{ each } i).$$

Proof. This is exactly the ${}_1F_0$ series with $h(\lambda,k)$ replaced using the lemma. □

Corollary 3.20 *For* $\lambda \in \mathcal{N}_N^P$*,* $\|\zeta_\lambda\|_p^2 = \frac{h(\lambda,k+1)}{h(\lambda,1)}$.

Proof. By the definition of the p-inner product,

$$F_k(x,y) = \sum_{\beta\in\mathcal{N}_N} \frac{1}{\|\zeta_\beta\|_p^2}\, \zeta_\beta(x)\zeta_\beta(y).$$

Set $y = 1_N$, and use $\|\zeta_\beta\|_p^2 = \mathcal{E}_+(\beta)\mathcal{E}_-(\beta)\|\zeta_\lambda\|_p^2$,

$$\zeta_\beta(1^N) = \mathcal{E}_-(\beta)\zeta_\lambda(1^N) = \mathcal{E}_-(\beta)(Nk+1)_\lambda/h(\lambda,1),$$

for $\lambda = \beta^+$. This shows

$$\prod_{i=1}^{N}(1-x_i)^{-(Nk+1)} = \sum_{\lambda\in\mathcal{N}_N^P} \frac{(Nk+1)_\lambda}{\|\zeta_\lambda\|_p^2 h(\lambda,1)} \sum_{\beta\in S_N\lambda} \frac{\zeta_\beta(x)}{\mathcal{E}_+(\beta)}$$

$$= \sum_{\lambda\in\mathcal{N}_N^P} \frac{(Nk+1)_\lambda}{\|\zeta_\lambda\|_p^2 h(\lambda,1)\mathcal{E}_+(\lambda^R)}\, j_\lambda(x).$$

Match up the coefficients with the ${}_1F_0$-expansion. □

The value of $\|\zeta_\lambda\|_p^2$ was first obtained by Sahi who used a recurrence relation (lowering the degree).

There is one more important S_N-invariant inner product for which $T_i x_i$ is self-adjoint, $1 \le i \le N$, obtained by considering polynomials as (analytic) functions on the torus in $\mathbb{C}^N$ (see [D3], Proposition 4.2).

Definition 3.21 *For f, g polynomials with coefficients in $\mathbb{Q}(k)$, let*

$$(3.4)\quad \langle f, g\rangle_{\mathbb{T}} := \frac{\Gamma(k+1)^N}{(2\pi)^N \Gamma(Nk+1)} \int_{\mathbb{T}^N} f(x) g^v(x) \left| \prod_{1\le j<\ell\le N} (x_j - x_\ell)(x_j^{-1} - x_\ell^{-1}) \right|^k dm(x),$$

where $g^v(x) := g(x_1^{-1}, x_2^{-1}, \ldots, x_N^{-1})$, $dm(x) = d\theta_1 \cdots d\theta_N$ and $x_j = \exp(\sqrt{-1}\,\theta_j)$, $-\pi < \theta_j \le \pi$, $1 \le j \le N$.

Beerends and Opdam [BO] evaluated $\langle J_\lambda, J_\lambda\rangle_{\mathbb{T}}$, from which one can deduce: For $\lambda \in \mathcal{N}_N^P$, $g \in E_\lambda$,

$$\|g\|_{\mathbb{T}}^2 = \frac{(Nk+1)_\lambda}{((N-1)k+1)_\lambda} \|g\|_p^2.$$

Baker and Forrester [BF3] computed $\|\zeta_\alpha\|_{\mathbb{T}}^2$ with a different method.

4 Subgroup alternating polynomials

Whenever a polynomial is skew-symmetric for a parabolic subgroup of S_N it is divisible by an appropriate minimal alternating polynomial (a product of discriminants). The evaluation of the quotient at $x = 1^N$ is a generalization of the Weyl dimension formula. This evaluation for polynomials in E_λ ($\lambda \in \mathcal{N}_N^P$) will be carried out in this section, by constructing for each interval $I \subset [1, N]$ a skew operator ψ_I with at least these properties:

(1) $\psi_I w = \operatorname{sgn}(w) w \psi_I$, for $w \in S_I$;

(2) $[\psi_I, U_j] = 0$ for $j \notin I$;

(3) if I_1 is an interval disjoint from I, then $[\psi_I, w] = 0$ for $w \in S_{I_1}$;

(4) if $\alpha \in \mathcal{N}_N$ and satisfies $(>, I)$, then ψ_I maps $\operatorname{span}\{\zeta_{w\alpha}\colon w \in S_I\}$ to itself.

Heuristically, one might suspect $\prod_{i<j}(\tau_i - \tau_j)$ works, but it is not skew because of non-commutativity, and $\prod_{i<j}(U_i - U_j)$ has the wrong transformation properties (for $(\#I) \ge 3$).

Definition 4.1 *For an interval I, let a_I denote the minimal alternating polynomial for I, that is, $a_I(x) := \Pi\{x_i - x_j :\ i < j \text{ and } i, j \in I\}$.*

Let $\mathcal{A}_I$ denote the associated division symmetrizing operator on polynomials:

$$\mathcal{A}_I f(x) := \frac{1}{(\#I)!} \sum_{w \in S_I} \operatorname{sgn}(w) f(xw)/a_I(x).$$

For α satisfying $(>, I)$ we will evaluate the functional $(\mathcal{A}_I a_{\alpha;I})(1^N)$, in fact, the more general situation for a collection of disjoint intervals $((\mathcal{A}_{I_1}\mathcal{A}_{I_2}\cdots)f)(1^N)$, for suitable f.

We will impose one more condition on ψ_I, which, surprisingly, is enough to determine the restriction to X uniquely. We will also construct an operator on X, using the Gram matrix H, satisfying the conditions; this will allow the determination of the matrix entries for ψ_I in the basis $\{g_{w\alpha} : w \in S_I\}$.

The aforementioned condition comes from the idea of a "reversing" transformation: the requirement that $\psi_I \zeta_{w\alpha}$ is a simultaneous eigenvector of $\tau_i - k\sum_{i<j\leq \ell+m}(ij)$, for $\ell < i \leq \ell + m$. Note that $U_i = \tau_i - k\sum_{\ell<j<i}(ij)$, so this reverses the interval I.

Definition 4.2 *For $i \in I$, let $\theta_i := \tau_i - \frac{k}{2}\sum_{j\neq i, j\in I}(ij)$.*

We show later that if ψ_I is skew and $[\psi_I, \theta_i] = 0$ for each $i \in I$, then ψ_I has the reversing property.

Fix α satisfying $(>, I)$, $I = [\ell+1, \ell+m]$ and $X = \mathrm{span}\{g_{w\alpha} : w \in S_I\} = \mathrm{span}\{\zeta_{w\alpha} : w \in S_I\}$.

For a linear transformation A on X we use the matrix notation $Ag_\beta = \sum_{\gamma\in S_I\alpha} A(\gamma, \beta) g_\gamma$.

Lemma 4.3 *The linear transformation A on X is skew if and only if $A(w_1\alpha, w_2\alpha) = sgn(w_1)c(w_1^{-1}w_2)$ for some function c on S_I.*

Proof. Given the function c define A as indicated (Recall $w_1 \neq w_2$ implies $w_1\alpha \neq w_2\alpha$). The transformation A is skew if and only if $(\mathrm{sgn}\ w_1)w_1 A g_{w_2\alpha} = Aw_1 g_{w_2\alpha}$ for each $w_1, w_2 \in S_I$, that is

$$\begin{aligned}\mathrm{sgn}(w_1)\sum_{w\in S_I} A(w\alpha, w_2\alpha)w_1 g_{w\alpha} &= \mathrm{sgn}(w_1)\sum_{w\in S_I} A(w_1^{-1}w\alpha, w_2\alpha) g_{w\alpha}\\ &= \sum_{w\in S_I} A(w\alpha, w_1w_2\alpha) g_{w\alpha}.\end{aligned}$$

Matching up coefficients of $g_{w\alpha}$ shows that A is skew if and only if

$$\mathrm{sgn}(w_1)A(w_1^{-1}w\alpha, w_2\alpha) = A(w\alpha, w_1w_2\alpha)$$

for all $w, w_1, w_2 \in S_I$, consistent with the relation

$$A(w\alpha, w_1w_2\alpha) = \mathrm{sgn}(w)c(w^{-1}w_1w_2). \qquad \square$$

Corollary 4.4 *With the same hypotheses,*

$$Aa_{\alpha;I} = (\sum_{w\in S_I} sgn(w)c(w))j_{\alpha;I}$$

and

$$Aj_{\alpha;I} = (\sum_{w\in S_I} c(w))a_{\alpha;I}.$$

Proof. By Definition 3.10,

$$\begin{aligned} Aa_{\alpha;I} &= \sum_{w_1}\sum_{w_2} \operatorname{sgn}(w_2) A(w_1\alpha_1 w_2\alpha) g_{w_1\alpha} \\ &= \sum_{w_1} g_{w_1\alpha}\left(\sum_{w_2} \operatorname{sgn}(w_2) c(w_1^{-1}w_2)\operatorname{sgn}(w_1)\right) \\ &= \sum_{w_1} g_{w_1\alpha}\left(\sum_{w_3} \operatorname{sgn}(w_3) c(w_3)\right), \end{aligned}$$

changing the second summation variable $w_2 = w_1 w_3$. A similar argument shows $Aj_{\alpha;I} = \left(\sum_w c(w)\right) a_{\alpha;I}$. $\square$

Two more relations apply when α satisfies $(>, I)$:

$$\kappa_i(w\alpha) = \kappa_{w^{-1}(i)}(\alpha), \quad \text{for } i \in I,\ w \in S_I; \tag{4.1}$$

$$\mathcal{E}_\epsilon(w\alpha; I)\mathcal{E}_\epsilon(\sigma_I w\alpha; I) = \mathcal{E}_\epsilon(\alpha^R; I),\ \text{for } w \in S_I,\ \epsilon = \pm. \tag{4.2}$$

For the second equation, note for any given $w\alpha$, and $\ell + 1 \le i < j \le \ell + m$, the term $1 + \frac{\epsilon k}{\kappa_i(\alpha)-\kappa_j(\alpha)}$ appears in $\mathcal{E}_\epsilon(w\alpha; I)$ if $w(j) < w(i)$, else in $\mathcal{E}_\epsilon(\sigma_I w\alpha; I)$; note $(w\alpha)_{w(i)} = \alpha_i$.

Lemma 4.5 *Suppose α satisfies $(>, I)$, then*

$$\theta_i g_\beta = \kappa_i'(\beta) g_\beta + \frac{k}{2} \sum_{j\in I, j\neq i} sgn(\beta_j - \beta_i)(ij) g_\beta,$$

for $\beta \in S_I\alpha$.

Proof. By Proposition 3.5,

$$\begin{aligned} \theta_i g_\beta &= \kappa_i'(\beta) g_\beta + \frac{k}{2}\sum\{(ij)g_\beta :\ j \in I,\ \beta_j > \beta_i\} \\ &\quad - \frac{k}{2}\sum\{(ij)g_\beta :\ j \in I,\ \beta_j < \beta_i\}. \end{aligned}$$

The case $\beta_i = \beta_j$ cannot occur. $\square$

The first important example of a skew operator commuting with each θ_i is defined using the Gram matrix for H; thus the domain is just the space X (for a given α satisfying $(>, I)$).

Let P be the operator on X with the matrix $P(w_1\alpha, w_2\alpha) = \operatorname{sgn}(w_1)\delta_{w_1,w_2}$ ("P" suggests parity). Note that $P^2 = 1$.

Proposition 4.6 *The operator on X with the matrix PH is skew and $[PH, \theta_i] = 0$ for $i \in I$.*

Proof. Let A_i be the matrix for θ_i in the basis $\{g_{w\alpha} : \ w \in S_I\}$. By the lemma, $A_i(\gamma,\beta) = \kappa_i'(\beta)$ if $\gamma=\beta$, and $= \frac{k}{2}\operatorname{sgn}(\beta_j-\beta_i)$ if $\gamma=(i)\beta$, and $=0$ else. Thus $A_i(\gamma,\beta)=0$ unless $\gamma=\beta$ or $\gamma=(ij)\beta$ for some $j\in I$, $j\neq i$. This shows $PA_iP = A_i^T$ (T for transpose), since

$$(PA_iP)(w_1\alpha, w_2\alpha) = \operatorname{sgn}(w_1)\operatorname{sgn}(w_2)A_i(w_1\alpha, w_2\alpha),$$

thus

$$(PA_iP)((ij)\beta,\beta) = -A_i((ij)\beta,\beta), \text{ for } \beta\in S_I\alpha.$$

Also θ_i is self-adjoint and H is a scalar multiple of the Gram matrix for $\{g_{w\alpha} : \ w\in S_I\}$, hence $A_i^TH = HA_i$, that is $PA_iPH = HA_i$ and $A_iPH = PHA_i$.

The operator PH is skew by the lemma (and $H(w_1\alpha, w_2\alpha) = H(\alpha, w_1^{-1}w_2\alpha)$, $w_1, w_2 \in S_I$). ☐

Proposition 4.7 *Suppose A is a skew operator on X, corresponding to the function c on S_I, then $[A,\theta_i]=0$ for each $i\in I$ if and only if*

$$(\kappa_i'(w\alpha)-\kappa_i'(\alpha))c(w) = \frac{k}{2}\sum_{j\in I, j\neq i} c((ij)w)(sgn(i-j)+sgn(w^{-1}(j)-w^{-1}(i))),$$

$$\text{for each } w\in S_I,\ i\in I.$$

Proof. For a linear transformation A on X the condition $[A,\theta_i]=0$ is equivalent to

$$\begin{aligned}&\kappa_i'(\beta)A(\gamma,\beta)+\frac{k}{2}\sum_{j\neq i}\operatorname{sgn}(\beta_j-\beta_i)A(\gamma,(ij)\beta)\\ &\quad = \kappa_i'(\gamma)A(\gamma,\beta)+\frac{k}{2}\sum_{j\neq i}\operatorname{sgn}(\gamma_i-\gamma_j)A((ij)\gamma,\beta),\end{aligned}$$

$(\gamma,\beta\in S_I\alpha, j\in I)$. Let $\gamma=w_1\alpha$, $\beta = w_2\alpha$ and replace $A(w_1\alpha,w_2\alpha)$ by $\operatorname{sgn}(w_1)c(w_1^{-1}w_2)$. The equation becomes

$$\begin{aligned}&(\kappa_i'(w_2\alpha)-\kappa_i'(w_1\alpha))\operatorname{sgn}(w_1)c(w_1^{-1}w_2)\\ &\quad = \frac{k}{2}\operatorname{sgn}(w_1)\sum_{j_0\neq i_0} c(w_1^{-1}(w_1(i_0),w_1(j_0))w_2)(\operatorname{sgn}(i_0-j_0)\\ &\qquad + \operatorname{sgn}(w_2^{-1}w_1(j_0)-w_2^{-1}w_1(i_0))),\end{aligned}$$

where $i_0 = w_1^{-1}(i)$, $j_0 = w_1^{-1}(j)$. Canceling out $\operatorname{sgn}(w_1)$ gives an equation depending only on $w := w_1^{-1}w_2$ and i_0, because $\kappa_i'(w_2\alpha) = \kappa_{w_1(i_0)}'(w_2\alpha) = \kappa_{i_0}'(w\alpha)$ and $\kappa_i'(w_1\alpha) = \kappa_{i_0}'(\alpha)$. This is the equation in the statement. Note for any $w\in S_I$, $i\in I$, $\kappa_i'(\alpha) = \kappa_{w(i)}'(w\alpha)$, and $\operatorname{sgn}(\alpha_i-\alpha_j) = -\operatorname{sgn}(j-i)$, for $i,j\in I$. ☐

Corollary 4.8 *If A is skew and $[A, \theta_i] = 0$ for each $i \in I$, and $k > 0$, then the values $c(w)$ are uniquely determined for given $c(1)$, $w \in S_I$.*

Proof. A certain subset of the equations in the proposition is extracted. For any $w \neq 1$ there is a unique $i \in I$ so that $w(j) = j$ for $j < i$ and $w^{-1}(i) > i$. Specialize the equations to these values of w, i; $\operatorname{sgn}(i-j) + \operatorname{sgn}(w^{-1}(j) - w^{-1}(i)) \neq 0$ exactly when $i < j$ and $w^{-1}(i) > w^{-1}(j)$ (by construction $j < i$ implies $w^{-1}(j) < w^{-1}(i)$).

Thus

$$(\kappa'_{w^{-1}(i)}(\alpha) - \kappa'_i(\alpha))c(w) = -k \sum \{c((ij)w) : \; w^{-1}(i) > w^{-1}(j), \; i < j \leq \ell + m\}.$$

The coefficient of $c(w)$ is nonzero, and if $\beta = (ij)w\alpha$ with $w^{-1}(i) > w^{-1}(j)$, then $\beta \succ w\alpha$. Thus $c(w)$ is uniquely determined in terms of the values $\{c(w_1) : w_1\alpha \succ w\alpha, \; w_1 \in S_I\}$ when $k \neq 0$. $\square$

This corollary shows that an operator on polynomials that is skew for S_I and commutes with θ_i, $i \in I$ is determined on each X ($= \operatorname{span}\{w\zeta_\alpha : \; w \in S_I\}$, α satisfies $(>, I)$) as a scalar multiple of PH. We derive some equations which will be instrumental in computing the multiple.

Proposition 4.9 *Suppose A is a skew linear operator on X and $[A, \theta_i] = 0$ for each $i \in I$, then there is a constant b such that $A = bPH$ and*

(1) $Aj_{\alpha;I} = b\mathcal{E}_+(\alpha^R; I)a_{\alpha;I}$;

(2) $Aa_{\alpha;I} = b\mathcal{E}_-(\alpha^R; I)j_{\alpha,I}$;

(3) $A\zeta_{w\alpha} = b \; sgn(w)\mathcal{E}_+(w\alpha; I)\sigma_I\zeta_{\sigma_I w\alpha}$, for $w \in S_I$;

(4) $A^2 = b^2\mathcal{E}_+(\alpha^R; I)\mathcal{E}_-(\alpha^R; I)1$.

Proof. The fact that A is a scalar multiple of PH follows from Corollary 4.8 and Proposition 4.6. Equations (1) and (2) follow from Corollary 3.16 and Lemma 4.4. We prove (3) by exhibiting a simultaneous eigenvector structure for $\{\sigma_I\zeta_{w\alpha} : \; w \in I\}$ and its relation to A.

For $i \in I$, let $U_i^R = \tau_i - k\sum_{i<j\leq \ell+m}(ij)$, then $\sigma_I U_{\sigma_I(i)}\sigma_I = U_i^R$ (note $\sigma_I(i) = 2m + \ell + 1 - i$). Further $U_i^R A = AU_i$, indeed $U_i = \theta_i + \frac{k}{2} \cdot \sum_{j\in I, j\neq i} \operatorname{sgn}(j-i)(ij)$ and $U_i^R = \theta_i - \frac{k}{2}\sum_{j\in I, j\neq i}\operatorname{sgn}(j-i)(ij)$, and $[A, \theta_i] = 0$, $A(ij) = -(ij)A$ by hypothesis. For $w \in S_I$, $\sigma_I A\zeta_{w\alpha}$ is an eigenvector of $U_{\sigma_I(i)}$ with eigenvalue $\kappa_i(w\alpha)$, each $i \in I$, because $\kappa_i(w\alpha)A\zeta_{w\alpha} = AU_i\zeta_{w\alpha} = U_i^R A\zeta_{w\alpha} = \sigma_I U_{\sigma_I(i)}(\sigma_I A\zeta_{w\alpha})$. Since $\kappa_i(w\alpha) = \kappa_{\sigma_I(i)}(\sigma_I w\alpha)$, this shows there is a constant $v(w)$ so that $\sigma_I A\zeta_{w\alpha} = v(w)\zeta_{\sigma_I w\alpha}$, each $w \in S_I$. We use (1) to find $v(w)$; on the one hand

$$\begin{aligned} Aa_{\alpha;I} &= b\mathcal{E}_-(\alpha^R; I)j_{\alpha;I} \\ &= b\mathcal{E}_-(\alpha^R; I)\sigma_I j_{\alpha;I} \\ &= b\mathcal{E}_-(\alpha^R; I)\mathcal{E}_+(\alpha^R; I)\sigma_I \sum_{w\in S_I} \frac{1}{\mathcal{E}_+(w\alpha; I)}\,\zeta_{w\alpha}; \end{aligned}$$

on the other hand

$$Aa_{\alpha;I} = \mathcal{E}_-(\alpha^R;I)\sum_{w\in S_I}\frac{\mathrm{sgn}(w)}{\mathcal{E}_-(w\alpha;I)}\ v(w)\sigma_I\zeta_{I_\sigma w\alpha}.$$

In the first equation, change the summation variable to $\sigma_I w$, and match up coefficients of $\sigma_I\sigma_{w\alpha}$ in the two equations; this shows

$$v(w) = b\ \mathrm{sgn}(w)\mathcal{E}_-(w\alpha;I)\mathcal{E}_+(\alpha^R;I)/\mathcal{E}_+(\sigma_I w\alpha;I) = b\ \mathrm{sgn}(w)\mathcal{E}_-(w\alpha;I)\mathcal{E}_+(w\alpha;I)$$

(by (4.2)). Finally,

$$\begin{aligned} A^2\zeta_{w\alpha} &= b^2\mathrm{sgn}(\sigma_I)v(w)\sigma_I A\zeta_{\sigma_I w\alpha} \\ &- b^2\mathrm{sgn}(\sigma_I)v(w)v(\sigma_I w)\zeta_{w\alpha} \\ &= b^2\mathcal{E}_+(\alpha^R;I)\mathcal{E}_-(\alpha^R;I)\zeta_{w\alpha},\ w\in S_I. \qquad \square \end{aligned}$$

We construct a skew operator commuting with each θ_i, $i\in I$, in the algebra generated by $\{\tau_i:\ i\in I\}\cup S_I$, by induction on the size of the interval.

Definition 4.10 *For the interval $I=[\ell+1,\ell+m]$ and $1\le s<m$, let $\psi_1 := 1$,*

$$\begin{aligned} \tilde{\psi}_{s+1} &:= U_{\ell+1}U_{\ell+2}\cdots U_{\ell+s}\psi_s; \\ \psi_{s+1} &:= \tilde{\psi}_{s+1} - \sum_{i=\ell+1}^{\ell+s}(i,\ell+s+1)\tilde{\psi}_{s+1}(i,\ell+s+1). \end{aligned}$$

Then $\psi_I := \psi_m$.

Theorem 4.11 *The operator ψ_I satisfies*

(1) $\psi_I w = sgn(w)w\psi_I$ for $w\in S_I$;

(2) $\psi_I w = w\psi_I$ for $w\in S_{[1,\ell]}\times S_{[\ell+m+1,N]}$;

(3) $[\psi_I,U_j]=0$ for $j\notin I$;

(4) $[\psi_I,\theta_i]=0$, for $i\in I$;

(5) for any α satisfying $(\geq,I)$, span$\{w\zeta_\alpha:\ w\in S_I\}$ is an invariant subspace of ψ_I.

Proof. Properties (2) and (3) follow immediately from the defintion; and property (5) is a consequence of (1). Let (1_s) be the condition $\psi_s(ij) = -(ij)\psi_s$ for $\ell+1 \leq i < j \leq \ell+s$, and (4_s) be

$$\left[\psi_s, \tau_i - \frac{k}{2}\sum_{j=\ell+1, j\neq i}^{\ell+s}(ij)\right] = 0.$$

The case $s = 1$ is trivial. Inductively, suppose (1_s) and (4_s) are true. For convenience, let $t = \ell+s+1$: then $(ij)\tilde{\psi}_{s+1} = -\tilde{\psi}_{s+1}(ij)$ for $\ell+1 \leq i < j \leq \ell+s$, because $[U_{\ell+1}\cdots U_{\ell+s}, (ij)] = 0$ (see Proposition 3.7). Now evaluate $(ij)\psi_{s+1}(ij)$ using the definition; each term in the sum is transformed to the negative of the corresponding term in ψ_{s+1}, with the exception of the terms labeled i and j which are also interchanged; note $(i,j)(i,t) = (j,t)(i,j)$. It suffices to show $(\ell+s,t)\psi_{s+1}(\ell+s,t) = -\psi_{s+1}$; again the terms in the sum correspond to the negatives, example: $(\ell+s,t)(i,t)\tilde{\psi}_{s+1}(i,t)(\ell+s,t) = (i,t)(i,\ell+s)\tilde{\psi}_{s+1}(i,\ell+s)(i,t) = -(i,t)\tilde{\psi}_{s+1}(i,t)$. The other part is $(\ell+s,t)(\tilde{\psi}_{s+1} - (\ell+s,t)\tilde{\psi}_{s+1}(\ell+s,t))(\ell+s,t)$. This shows (1_{s+1}).

To show (4_{s+1}), let $B_j := \tau_j - \frac{k}{2}\sum_{i=\ell+1,i\neq j}^{\ell+s}(ij)$, for $\ell+1 \leq j \leq \ell+s+1$. Then for $j \leq \ell+s$, $[\tilde{\psi}_{s+1}, B_j] = 0$ by (4_s) and $[U_{\ell+1}\cdots U_{\ell+s}, B_j] = 0$; the latter follows from $\tau_j = (\ell+1,j)U_{\ell+1}(\ell+1,j)$ and $U_{\ell+1}\cdots U_{\ell+s}$ commutes with $U_{\ell+1}$ and $w \in S_{[\ell+1,\ell+s]}$.

It will suffice to show $[\psi_{s+1}, B_t] = 0$ since

$$(j,t)B_t(j,t) = \tau_j - \frac{k}{2}\sum_{i=\ell+1,i\neq j}^{\ell+s+1}(ij).$$

For $\ell+1 \leq i \leq \ell+s$,

$$\begin{aligned}
[(i,t)\tilde{\psi}_{s+1}(i,t), B_t] &= (i,t)[\tilde{\psi}_{s+1}, (i,t)B_t(i,t)](i,t) \\
&= (i,t)\left[\tilde{\psi}_{s+1}, B_i - \frac{k}{2}(i,t)\right](i,t) \\
&= -\frac{k}{2}(i,t)[\tilde{\psi}_{s+1}, (i,t)](i,t) \\
&= \frac{k}{2}[\tilde{\psi}_{s+1}, (it)].
\end{aligned}$$

Also $[\tilde{\psi}_{s+1}, U_t] = 0$ by (3), that is,

$$\left[\tilde{\psi}_{s+1}, \tau_t - k\sum_{i=\ell+1}^{\ell+s}(i,t)\right] = 0,$$

equivalently,

$$[\tilde{\psi}_{s+1}, B_t] = \frac{k}{2}\sum_{i=\ell+1}^{\ell+s}[\tilde{\psi}_{s+1}, (it)].$$

This shows

$$\left[\tilde{\psi}_{s+1}-\sum_{i=\ell+1}^{\ell+s}(i,t)\tilde{\psi}_{s+1}(i,t),B_t\right]=0. \qquad \square$$

Theorem 4.12 *Suppose α satisfies $(>,I)$, then*

$$\psi_I|X=\Pi\{\kappa_i(\alpha)-\kappa_j(\alpha):\ \ell+1\leq i<j\leq \ell+m\}PH.$$

Proof. By 4.9, $PH\zeta_\alpha=\sigma_I\zeta_{\sigma_I\alpha}$; thus it suffices to compute the coefficient of $\zeta_{\sigma_I\alpha}$ in $\sigma_I\psi_I\zeta_\alpha$. We use the inductive framework from Definition 4.10 and Theorem 4.11. For fixed $s<m$, let σ_0,σ_1 be the reversing permutations for the intervals $[\ell+1,\ell+s]$, $[\ell+1,\ell+s+1]$ respectively. The inductive hypothesis is that

$$\psi_s\zeta_\gamma=\prod_{\ell+1\leq i<j\leq\ell+s}(\kappa_i(\gamma)-\kappa_j(\gamma))\sigma_0\zeta_{\sigma_0\gamma}$$

for any γ satisfying $(>,[\ell+1,\ell+s])$ (trivial for $s=1$). Fix α satisfying $(>,I)$; let

$$\pi_r:=\prod\{\kappa_i(\alpha)-\kappa_j(\alpha):\ \ell+1\leq i<j\leq\ell+s+1,\ i\neq r,\ j\neq r\},$$

and

$$\pi'_r:=\prod\{\kappa_i(\alpha):\ \ell+1\leq i\leq\ell+s+1,\ i\neq r\}.$$

For $\ell+1\leq j\leq\ell+s+1$, let $w_{(j)}$ be the cycle $(\ell+s+1,\ell+s,\ldots,j+1,j)$ in $S_{[\ell+1,\ell+s+1]}$. In the notation of Definition 4.10,

$$\psi_{s+1}=\sum_{j=\ell+1}^{\ell+s+1}(-1)^{\ell+s+1-j}w_{(j)}^{-1}\tilde{\psi}_{s+1}w_{(j)},$$

because $w_{(j)}=(\ell+s,\ell+s-1,\ldots,j)(j,\ell+1+s)$ (in cycle notation) and $\operatorname{sgn}(w_{(j)})=\ell+s+1-j$; coming from the skew property of $\tilde{\psi}_{s+1}$ for $S_{[\ell+1,\ell+s]}$.

Let $\alpha_{(j)}=w_{(j)}\alpha$; thus

$$\alpha_{(j)}=(\alpha_1,\ldots,\alpha_{\ell+1},\alpha_{\ell+2},\ldots,\alpha_{j-1},\alpha_{j+1},\ldots,\alpha_{\ell+s+1},\alpha_j,\alpha_{\ell+s+2},\ldots),$$

which satisfies $(>,[\ell+1,\ell+s])$. We claim the coefficient of $\zeta_{\sigma_1\alpha}$ in $\sigma_1 w_{(j)}^{-1}\tilde{\psi}_{s+1}w_{(j)}\zeta_\alpha$ is $\pi_j\pi'_j$. By a standard identity for alternating polynomials

$$\sum_{j=\ell+1}^{\ell+s+1}(-1)^{\ell+s+1-j}\pi_j\pi'_j=\prod_{\ell+1\leq i<j\leq\ell+s+1}(\kappa_i(\alpha)-\kappa_j(\alpha)).$$

To establish the claim:

$$w_{(j)}\zeta_\alpha=\zeta_{\alpha_{(j)}}+\sum\left\{b(\gamma)\zeta_\gamma:\ \gamma\in S_{[\ell+1,\ell+s+1]}\alpha \text{ and } \gamma\succ\alpha_{(j)}\right\}$$

(the triangularity of the matrix relating the bases $\{\zeta_{w\alpha}\}$ and $\{w\zeta_\alpha\}$). By the inductive hypothesis and

$$\sigma_1 w_{(j)}^{-1}\tilde{\psi}_{s+1}w_{(j)}\zeta_\alpha = \sigma_1 w_{(j)}^{-1}\sigma_0(\pi_j\pi_j'\zeta_{\sigma_0\alpha_{(j)}} + \sum\{b'(\gamma)\zeta_{\sigma_0\gamma} :\ \gamma \succ \alpha_{(j)}\}),$$

for some coefficients $b'(\gamma)$. But

$$\sigma_1 w_{(j)}^{-1}\sigma_0 = w_{(\sigma_1 j)}^{-1} \qquad (\sigma_1(j) = 2\ell + s + 2 - j),$$

which fixes $[\ell+1, 2\ell+s+1-j]$ pointwise, which shows that

$$\sigma_1 w_{(j)}^{-1}\sigma_0\zeta_{\sigma_0\gamma} \in \operatorname{span}\{\zeta_{w\sigma_0\gamma} :\ w \in S_{[2\ell+s+2-j,\ell+s+1]}\}.$$

Now $(\sigma_0\gamma)_{\ell+s+1} = \gamma_{\ell+s+1} \in \{\alpha_{j+1},\ldots,\alpha_{\ell+s+1}\}$, because $\gamma \succ w_{(j)}\alpha$; hence for any $\beta = w\sigma_0\gamma$ in this span, there must be an element of $\{\alpha_{j+1},\ldots,\alpha_{\ell+s+1}\}$ not appearing in $\{\beta_{\ell+1},\ldots,\beta_{2\ell+s+1-j}\}$, thus $\beta \neq \sigma_1\alpha$.

The argument will be finished once it is shown that the coefficient of $\zeta_{\sigma_1\alpha}$ in the $\{\zeta_{w\alpha}\}$ expansion of $w^{-1}\zeta_{w\sigma_1\alpha}$ is 1: in the notation of (3.1),

$$\begin{aligned} w^{-1}\zeta_{w\sigma_1\alpha} &= w^{-1}\left(g_{w\sigma_1\alpha} + \sum_{\gamma\succ w\sigma_1\alpha} B(\gamma, w\sigma_1\alpha)g_\gamma\right) \\ &= g_{\sigma_1\alpha} + \sum_\gamma B(\gamma, w\sigma_1\alpha)g_{w^{-1}\gamma}; \end{aligned}$$

$\gamma \succ w\sigma_1\alpha$ implies $w^{-1}\gamma \neq \sigma_1\alpha$, the minimality of $\sigma_1\alpha$ shows the coefficient of $\zeta_{\sigma_1\alpha}$ in the right-hand side is 1. $\square$

Corollary 4.13

$$\psi_I a_{\alpha;I} = \prod_{\ell+1\le i<j\le \ell+m}(\kappa_i(\alpha) - \kappa_j(\alpha) - k)j_{\alpha,I}.$$

Proof. $\prod_{i<j}(\kappa_i(\alpha) - \kappa_j(\alpha))\mathcal{E}_-(\alpha^R; I)$ has the specified value; see Lemma 3.13. $\square$

We return to the problem of evaluating $\mathcal{A}_I f(1^N)$. The proof of the following comes later.

Theorem 4.14 *Suppose $\{I_1, I_2, \ldots, I_t\}$ is a collection of disjoint subintervals of $[1,N]$, $m_i := \#I_i$, $1 \le i \le t$ and f is a polynomial, then*

$$(\mathcal{A}_{I_1}\mathcal{A}_{I_2}\cdots A_{I_t}f)(1^N) = \frac{1}{(Nk+1)_\mu \prod_{i=1}^t m_i!}\,(\psi_{I_1}\psi_{I_2}\cdots\psi_{I_t}f)(1^N),$$

where $\mu = (m_1-1, m_1-2, \ldots, 1, 0, m_2-1, \ldots, 0, \ldots, m_t-1, \ldots, 1, 0, \ldots)^+$.

In the sequel, for an operator A on polynomials, A^* denotes the adjoint with respect to the p-inner product.

Let $\iota(x) := \prod_{i=1}^N (1-x_i)^{-(Nk+1)}$, then $\langle f, \iota\rangle_p = f(1^N)$ for any polynomial f; more generally, $\langle f, F_k(\,\cdot\,, z)\rangle_p = f(z)$. Let $u_i := x_i/(1-x_i),\ 1 \le i \le N$.

Lemma 4.15

$$\mathcal{A}_{I_1}^* \cdots \mathcal{A}_{I_t}^* \iota(x) = \prod_{i=1}^t \left(\frac{1}{m_i!}\ a_{I_i}(u)\right) \prod_{j=1}^N (1-x_j)^{-(Nk+1)}.$$

Proof. First apply $\mathcal{A}_{I_j}$ to $F_k(x,z)$ with respect to z. Without loss of generality, assume $I_j = [1,m]$ (with $m = m_j$); then

$$\begin{aligned}
\mathcal{A}_{I_j} F_k(x,z) &= \frac{1}{m!}\left(\sum_{w \in S_{[1,m]}} \frac{\operatorname{sgn}(w)}{\prod_{i=1}^m (1 - x_i (zw)_i)}\right) \prod_{j=m+1}^N (1-x_j z_j)^{-1} \frac{\prod_{i,j=1}^N (1-x_i z_j)^{-k}}{a_{[1,m]}(z)} \\
&= \frac{1}{m!} a_{[1,m]}(x) \prod_{i,j=1}^m (1-x_i z_j)^{-1} \prod_{j=m+1}^N (1-x_j z_j)^{-1} \prod_{i,j=1}^N (1-x_i z_j)^{-k},
\end{aligned}$$

where the sum evaluates to $a_{[1,m]}(x) a_{[1,m]}(z) / \prod_{i,j=1}^m (1 - x_i z_j)$ by an identity of Cauchy (this identity was used by Baker and Forrester [BF4] in a similar calculation). Because the intervals are disjoint,

$$\mathcal{A}_{I_1} \mathcal{A}_{I_2} \cdots \mathcal{A}_{I_t} F_k(x,z) = \prod_{r=1}^t h_r(x,z) F_k(x,z),$$

where

$$h_r(x,z) = \frac{1}{m_r!}\ a_{I_r}(x) \prod \{(1-x_i z_j)^{-1} :\ i,j \in I_r,\ i \neq j\}.$$

To find $\mathcal{A}_{I_1}^* \cdots \mathcal{A}_{I_t}^* \iota$ put $z = 1^N$ in this formula; note

$$h_r(x, 1^N) = \frac{1}{m_r!} a_{I_r}(x) \prod_{i \in I_r} (1-x_i)^{-(m_i - 1)} = \frac{1}{m_r!} a_{I_r}(u).$$

The operators ψ_I are generated by $\{T_i \rho_i :\ i \in I\}$ and transpositions. Recall $T_i \rho_i = T_i x_i + k$. Write T_i^u to denote action in the variable $(u_1, u_2, \ldots)$.

Lemma 4.16 *Suppose f is a polynomial in u, then*

$$(T_i x_i + k)(f(u)\iota(x)) = ((1+u_i)(T_i^u u_i + k) f(u))\iota(x),\ 1 \le i \le N.$$

Proof. The product rule $T_i(h(x)\iota(x)) = (T_ih(x))\iota(x) + h(x)\frac{\partial\iota(x)}{\partial x_i}$ applies because $\iota(x)$ is S_N-invariant. The chain rule implies

$$\frac{[(T_ix_i+k)(f(u)\iota(x))]}{\iota(x)} = f(u)(1+k) + (Nk+1)u_if(u) + u_i(1+u_i)\frac{\partial f(u)}{\partial u_i}$$
$$+ k\sum_{j\neq i}\frac{u_i(1+u_j)f(u) - u_j(1+u_i)f(u(ij))}{u_i(1+u_j) - u_j(1+u_i)}$$

(note $x_i = u_i/(1+u_i)$). The typical term in the sum equals

$$(1+u_i)\left(\frac{u_if(u) - u_jf(u(ij))}{u_i - u_j}\right) - u_if(u). \qquad \square$$

The effect on f is to raise the degree by 1; in fact, the highest degree term of the operator coincides with $T_i^{u*} = u_i(T_i^u u_i + k)$ ([D3], Proposition 4.1).

Lemma 4.17 *For an interval $I = [\ell+1, \ell+m]$ and a polynomial $f(u)$ of degree t,*

$$\psi_I^*(f(u)\iota(x)) = \left[\prod_{\ell+1\leq i<j\leq\ell+m}(T_i^* - T_j^*)f(u) + f_1(u)\right]\iota(x),$$

where $f_1(u)$ is a polynomial of degree $< t + m(m-1)/2$.

Proof. Using the inductive framework from Theorem 4.11, suppose the statement is true for the interval $[\ell+1, \ell+s]$ (trivial for $s=1$). In the notation of Definition 4.10, $\tilde{\psi}_{s+1}^* = \psi_s^*(U_{\ell+1}\cdots U_{\ell+s})^*$ and by Lemma 4.16,

$$U_i^*(f(u)\iota(x)) = ((1+u_i)(T_i^u u_i + k)f(u) - k\sum_{j<i}f(u(ij)))\iota(x),$$

which has the same highest degree terms as $(T_i^*f(u))\iota(x)$. By the commutativity of $\{T_i^*\}$, the highest degree term of $\tilde{\psi}_{s+1}^*(f(u)\iota(x))$ is

$$\left(\prod_{i=\ell+1}^{\ell+s}T_i^*\prod_{\ell+1\leq i<j\leq\ell+s}(T_i^* - T_j^*)f(u)\right)\iota(x)$$

(inductive hypothesis). Finally

$$\psi_{s+1}^* = \tilde{\psi}_{s+1}^* - \sum_{i=\ell+1}^{\ell+s}(i, \ell+s+1)\tilde{\psi}_{s+1}^*(i, \ell+s+1);$$

and the usual identity for $a_{[\ell+1,\ell+s+1]}$ finishes the proof; since

$$(i, \ell+s+1)T_i^*(i, \ell+s+1) = T_{\ell+s+1}^*. \qquad \square$$

Lemma 4.18
$$\left(\prod_{i=1}^t \psi_{I_i}^*\right)\iota(x) = c_I \prod_{i=1}^t a_{I_i}(u)\iota(x)$$
for some constant c_I*.*

Proof. By the previous lemma,

$$\frac{\left(\prod_{i=1}^t \psi_{I_i}^*\right)\iota(x)}{\iota(x)} = f_0(u) + f_1(u),$$

where $f_0(u) = \prod_{i=1}^t a_{I_i}(T^*)1$ and deg $f_1 < \sum_{i=1}^t m_i(m_i-1)/2$. Also $f_0(u)+f_1(u)$ is skew for $\prod_{i=1}^t S_{I_i}$ (direct product) by the skew property of ψ_I; which implies $f_1 = 0$ and $f_0(u) = c_I \prod_{i=1}^t a_{I_i}(u)$ because this is the unique skew polynomial of minimum degree. □

Proof of Theorem 4.14. The lemmas show that

$$\prod_{i=1}^t \psi_{I_i}^*\iota(x) = \left(\left(\prod_{i=1}^t a_{I_i}(T^{u*})\right)1\right)\iota(x).$$

In ([D3], Theorem 3.1) it was shown that $\prod_i a_{I_i} \in E_\mu$ for $\mu = (m_1-1, m_1-2, \ldots, 1, 0, m_2-1, \ldots, 1, 0, \ldots)^+$, thus (by Corollary 2.6)

$$\prod_{i=1}^t a_{I_i}(T^{u*})1 = (Nk+1)_\mu \prod_{i=1}^t a_{I_i}(u)$$

(see also [DH]); and so

$$\prod_{i=1}^t \psi_{I_i}^*\iota = (Nk+1)_\mu \left(\prod_{i=1}^t m_i!\right)\prod_{i=1}^t \mathcal{A}_{I_i}^*\iota.$$

Corollary 4.19 *Suppose* α *satisfies* $(>, I_i)$ *for each* i, $1 \le i \le t$, *then*

$$\left(\prod_{i=1}^t \mathcal{A}_{I_i} \sum_{w\in S_I} (sgn\ w) w\zeta_\alpha\right)(1^N)$$

$$= \frac{(Nk+1)_{\alpha^+}}{(Nk+1)_\mu}\,\frac{\mathcal{E}_-(\alpha,[1,N])}{h(\alpha^+,1)} \prod_{r=1}^t \prod\{\kappa_i(\alpha) - \kappa_j(\alpha) - k :\ i,j \in I_r,\ i<j\},$$

where $S_I := S_{I_1} \times S_{I_2} \cdots \times S_{I_t}$*.*

Proof. Let $\pi_r = \prod\{\kappa_i(\alpha) - \kappa_j(\alpha) - k : \ i, j \in I_r,\ i < j\}$. For each interval I_r, $\psi_{I_r} \sum_{w\in S_{I_r}} \operatorname{sgn}(w) w\zeta_\alpha = \pi_r \sum_{w\in S_{I_r}} w\zeta_\alpha$, by Corollary 4.13. Since $[\psi_{I_r}, w] = 0$ for $w \in S_{I_i}$ and $[\psi_{I_r}, \psi_{I_i}] = 0$, for $i \neq r$ (Theorem 4.11),

$$\prod_{r=1}^{t} \psi_{I_r} \sum_{w\in S_{I_r}} \operatorname{sgn}(w) w\zeta_\alpha = \prod_{r=1}^{t} \pi_r \sum_{w\in S_I} w\zeta_\alpha$$

(the sum is a product of sums over $S_{I_1}, \ldots, S_{I_t}$). Also

$$\sum_{w\in S_I} w\zeta_\alpha(1^N) = \left(\prod_{i=1}^{t} m_i!\right) \zeta_\alpha(1^N),$$

and

$$\zeta_\alpha(1^N) = (Nk+1)_\alpha \mathcal{E}_-(\alpha, [1,N])/h(\alpha^+, 1)$$

(by (3.2)). Further

$$\mathcal{A}_{I_r} \sum_{w\in S_{I_r}} \operatorname{sgn}(w) w\zeta_\alpha = \sum_{w\in S_{I_r}} \operatorname{sgn}(w) w\zeta_\alpha / a_{I_r}$$

(and this is a typical term in the product sum over $S_{I_1} \times \cdots \times S_{I_t}$). $\square$

5 Orthogonal Polynomials of Type B_N

This section deals with operators and orthogonal decompositions associated with the group generated by sign-changes and permutations of coordinates. Previously, Baker and Forrester [BF2] considered some of the orthogonal polynomials, called generalized Laguerre polynomials. These come from polynomials which are even in each coordinate or odd in each coordinate. The general situation is developed in the sequel. One consequence is a complete set of eigenfunctions for the Hamiltonian of the B_N spin Calogero model ($1/r^2$ interactions confined in harmonic potential), with arbitrary parity, that is, for any subset $A \subset [1,N]$ we find the eigenfunctions which are odd in x_i, $i \in A$ and even in x_i, $i \notin A$.

The underlying symmetry group W_N is the Weyl group of type B_N, called the hyperoctahedral group. It is generated by permutations of coordinates and sign-changes on $\mathbb{R}^N$. The reflections in W_N are $\{\sigma_{ij}, \tau_{ij} : \ 1 \le i < j \le N\}$ and $\{\sigma_i : \ 1 \le i \le N\}$, defined by $x\sigma_{ij} = (x_1, \ldots \overset{i}{x_j}, \ldots, \overset{j}{x_i}, \ldots)$, $x\tau_{ij} = (x_1, \ldots, -\overset{i}{x_j}, \ldots, -\overset{j}{x_i}, \ldots)$, $x\sigma_i = (x_1, \ldots, -\overset{i}{x_i}, \ldots)$. There are two parameters k, k_1 in the algebra of differential-difference operators:

$$T_i^B f(x) := \frac{\partial f}{\partial x_i} + k_1 \frac{f(x) - f(x\sigma_i)}{x_i} + k \sum_{j\neq i} \left\{ \frac{f(x) - f(x\sigma_{ij})}{x_i - x_j} + \frac{f(x) - f(x\tau_{ij})}{x_i + x_j} \right\} \tag{5.1}$$

$(1 \leq i \leq N$, for convenience $\sigma_{ij} = \sigma_{ji}$, $\tau_{ij} = \tau_{ji}$ for $j < i)$.

The S_N-theory can be applied to the analysis of W_N by writing polynomials in the form $x_A g(x_1^2, x_2^2, \ldots, x_N^2)$ where $x_A := \prod_{i\in A} x_i$, $A \subset [1, N]$. For a composition α let $\hat{p}_\alpha(x) := x_A p_\beta(x_1^2, \ldots, x_N^2)$ where $A = \{i : \alpha_i \text{ is odd}\}$, and $\beta_i = \lfloor \alpha_i/2 \rfloor$, $1 \leq i \leq N$. Observe

$$\sum_\alpha \hat{p}_\alpha(x) z^\alpha = \prod_{i=1}^N ((1 - x_i z_i)^{-1} \prod_{j=1}^N (1 - x_i^2 z_j^2)^{-k}).$$

The raising operator $\hat{\rho}_i$ is defined by $\hat{\rho}_i \hat{p}_\alpha = \hat{p}(\alpha_1, \ldots, \alpha_{i+1}, \ldots)$.

We will use $y \in \mathbb{R}^N$ to denote $(x_1^2, x_2^2, \ldots, x_N^2)$ and the A-type operators T_i act on y. The following properties hold ([D4], Prop. 2.1), for $A \subset [1, N]$, any polynomial g in y:

$$(5.2)\quad (\sigma_{ij} + \tau_{ij}) x_A g(y) = \begin{cases} (2x_A)(ij)g(y) & i, j \in A \text{ or } i, j \notin A, \\ 0 & \text{else}; \end{cases}$$

$$(5.3)\quad T_i^B \hat{\rho}_i (x_A g(y)) = \begin{cases} 2x_A(T_i \rho_i g(y)), & i \in A, \\ 2x_A((k_1 - k - \frac{1}{2})g(y) + T_i \rho_i g(y) - k \sum_{s\in A}(ij)g(y)), & i \notin A. \end{cases}$$

Definition 5.1 *For $1 \leq i \leq N$, $U_i^B := T_i^B \hat{\rho}_i - k \sum_{j<i}(\sigma_{ij} + \tau_{ij})$ acts on polynomials in x. For a subset $A \subset [1, N]$, let*

$$U_{A,i} := T_i \rho_i - k \sum \{(ij) : \ j < i \text{ and } j \in A\} \text{ for } i \in A,$$

$$U_{A,i} := T_i \rho_i + (k_1 - k - \frac{1}{2})1 - k \sum \{(ij) : \ j \in A \text{ or } (j \notin A \text{ and } j < i)\} \text{ for } i \notin A,$$

acting on polynomials in y.

Proposition 5.2 *For $A \subset [1, N]$, $U_i^B(x_A g(y)) = 2x_A U_{A,i} g(y)$, $1 \leq i \leq N$. Also*

$$\prod_{i\in A} T_i^B (x_A g(y)) = 2^{\#A} \prod_{i\in A} \left(U_{A,i} + k_1 - k - \frac{1}{2} \right) g(y).$$

This proves the commutativity of $\{U_i^B\}$ in terms of propositions about S_N; a direct proof is also possible.

Let V^B denote the intertwining operator for W_N, thus $T_i^B V^B = V^B \frac{\partial}{\partial x_i}$, and V^B is homogeneous of degree 0; also let ξ^B be the linear operator on polynomials defined by $\xi^B \hat{p}_\alpha = x^\alpha / \alpha!$. Here is a description of the eigenspace decomposition of $V^B \xi^B$ constructed in [D4]. Each space is irreducible for the algebra generated by $\{T_i^B \hat{\rho}_i : \ 1 \leq i \leq N\}$.

Definition 5.3 *A B-partition α is a composition $(\alpha_1, \alpha_2, \ldots, \alpha_N)$ where the odd and even parts are respectively nonincreasing, that is, $i < j$ and $\alpha_i \equiv \alpha_j \bmod 2$ implies $\alpha_i \geq \alpha_j$. A standard B-partition is one in which the odd parts come first (for some ℓ, α_i is odd for $i \leq \ell$, even for $i > \ell$). For $\alpha \in \mathcal{N}_N$, let*

$$h(\alpha) = (\lfloor \alpha_1/2 \rfloor, \lfloor \alpha_2/2 \rfloor, \ldots),\ b(\alpha) = (\alpha_1 - \lfloor \alpha_1/2 \rfloor, \alpha_2 - \lfloor \alpha_2/2 \rfloor, \ldots).$$

This is an example of a B-partition: $\alpha = (4,5,3,4,2,0,1)$, and $h(\alpha) = (2,2,1,2,1,0,0)$, $b(\alpha) = (2,3,2,2,1,0,1)$. The corresponding standard B-partition is $(5,3,1,4,4,2,0)$.

Any B-partition can be rearranged to a standard one; if α is a B-partition with ℓ odd parts, then there exists a standard B-partition $\tilde{\alpha}$ and $w \in S_N$ so that $w\tilde{\alpha} = \alpha$ and $1 \leq i < j \leq \ell$ or $\ell + 1 \leq i < j \leq N$ implies $w(i) < w(j)$ (note $\alpha_{w(i)} = \tilde{\alpha}_i$, each i).

The property of permutations described above will be used several times in the sequel.

Definition 5.4 *For any subset $A \subset [1, N]$ let $w_A \in S_N$ be the unique permutation satisfying: $w_A([1,\ell]) = A \quad$ (for $\ell = \#A$), and $1 \leq i < j \leq \ell$ or $\ell + 1 \leq i < j \leq N$ implies $w_A(i) < w_A(j)$. If $1 \leq \ell < N$, the correspondence $A \to w_A$ is one-to-one.*

Proposition 5.5 *Let $A \subset [1, N]$, $\ell = \#A$, then*

$$\begin{aligned} U_i^B w_A(x_1 x_2 \cdots x_\ell g(y)) &= w_A(2x_1 \cdots x_\ell U_{[1,\ell],s} g(y)) \\ &= w_A(U_s^B(x_1 \cdots x_\ell g(y))),\ \text{for } 1 \leq i \leq N,\ s = w^{-1}(i). \end{aligned}$$

Proof. $(T_i^B \hat{\rho}_i) w_A = w_A T_s^B \hat{\rho}_s$, for $s = w_A^{-1}(i)$ and $(ij) w_A = w_A(w_A^{-1}(i), w_A^{-1}(j))$ for $j \neq i$. The order preserving properties of w_A and Proposition 5.5 imply the stated equations. $\square$

The proposition shows how to find joint eigenfunctions of $\{U_i^B\}$ corresponding to arbitrary B-partitions once the standard case is done.

For a given standard B-partition α, with α_i being odd exactly when $1 \leq i \leq \ell$, let $G_\ell = S_{[1,\ell]} \times S_{[\ell+1,N]}$, $\beta = h(\alpha)$, $\mu = \beta^+$. Define

$$E_\alpha^B = \operatorname{span}\{x_1 x_2 \cdots x_\ell \zeta_\gamma(y) : \gamma = w\beta,\ w \in G_\ell\}.$$

Proposition 5.6 *E_α^B is invariant under G_ℓ, $U_i^B(x_1 \cdots x_\ell \zeta_\gamma(y)) = c_i x_1 \cdots x_\ell \zeta_\gamma(y)$, where $c_i = 2\kappa_i(\gamma)$ for $i \in [1,\ell]$, $c_i = 2(\kappa_i(\gamma) + k_1 - k - \frac{1}{2})$ for $i \in [\ell+1, N]$.*

This follows from Proposition 5.2. To express the eigenvalues of $V^B \xi^B$, let

$$\Lambda(\alpha) := (Nk+1)_{h(\alpha)^+}((N-1)k + k_1 + \frac{1}{2})_{b(\alpha)^+},$$

for $\alpha \in \mathcal{N}_N$. Then E_α^B is an eigenspace for $V^B \xi^B$ with eigenvalue $2^{-|\alpha|} \Lambda(\alpha)^{-1}$. It was shown in ([D4], Proposition 4.2) that $x_1 \cdots x_\ell \zeta_\beta(y)$ is an eigenfunction of $V^B \xi^B$ with this eigenvalue, and $V^B \xi^B$ commutes with $w \in W_N$.

Again there are three inner products in which each U_i^B is self-adjoint, and which are W_N-invariant: (1) the p-product, $\langle x^\alpha, \hat{p}_\beta\rangle_p = \delta_{\alpha\beta}$; (2) the B-product, $\langle f,g\rangle_B = f(T^B)g(x)|_{x=0}$ (depends on both parameters); (3) the N-torus product,

$$\langle f,g\rangle_{\mathbb{T}} := c_k \int_{\mathbb{T}^N} f(x)\check{g}(x) \prod_{1\le j<\ell\le N} |(x_j^2 - x_\ell^2)(x_j^{-2} - x_\ell^{-2})|^k dm(x)$$

(same notation as in 3.20; c_k is the normalizing constant). It is obvious that $\langle x_{A_1} g_1(y), x_{A_2} g_2(y)\rangle = 0$ if $A_1, A_2 \subset [1,N]$ and $A_1 \neq A_2$, by the W_N-invariance (change the sign of x_i for $i \in A_1 \setminus A_2$ or $i \in A_2 \setminus A_1$). Further $\langle x_A g_1(y), x_A g_2(y)\rangle = \langle g_1, g_2\rangle$ for the p- or $\mathbb{T}$-products, using the type A definition on the right.

By the results in Section 3, let $f(x) = x_1 \cdots x_\ell \zeta_\gamma(y) \in E_\alpha^B$, then

$$\|f\|_p^2 = \|\zeta_\gamma\|_p^2 = \mathcal{E}_+(\gamma)\mathcal{E}_-(\gamma)h(\gamma^+, k+1)/h(\gamma^+,1).$$

By the same argument as Proposition 2.5,

$$\|f\|_B^2 = f(T^B)f(x) = 2^{|\alpha|}\Lambda(\alpha)\|\zeta_\gamma\|_p^2. \tag{5.4}$$

In [D2] we showed that

$$\langle f,g\rangle_B = c_{k,k'} \int_{\mathbb{R}^N} \left(e^{-L/2}f\right)\left(e^{-L/2}g\right) \prod_{j=1}^N |x_j|^{2k_1} \prod_{1\le i<j\le N} |x_i^2 - x_j^2|^{2k} e^{-|x|^2/2} dx, \tag{5.5}$$

for polynomials f,g where $L := \sum_{i=1}^N (T_i^B)^2$ (the normalizing constant $c_{k,k'}$ is chosen to make $\langle 1,1\rangle_B = 1$ and is computed by means of the Macdonald-Selberg integral). Define a generalized Hermite polynomial

$$H_\beta(x) := e^{-L/2}(x_1 \cdots x_\ell \zeta_\gamma(y)), \tag{5.6}$$

with

$$\beta = (2\gamma_1 + 1, \ldots, 2\gamma_\ell + 1, 2\gamma_{\ell+1}, \ldots, 2\gamma_N) \quad (\text{so } \gamma = h(\beta)).$$

The complete orthogonal basis is given by $\{w_A H_\beta(x) : \ A \subset [1,N], \ \ell = \#A, \ \beta_i \text{ is odd exactly for } i \in [1,\ell]\}$.

By using the "non-symmetric" binomial coefficients introduced by Baker and Forrester [BF3] we can produce an expression for H_β in terms of $\{\zeta_\gamma\}$; although very little is known about the coefficients.

Definition 5.7 *For $\alpha \in \mathcal{N}_N$, the binomial coefficients (depending on k) are implicitly defined by*

$$\frac{\zeta_\alpha(y + 1^N)}{\zeta_\alpha(1^N)} = \sum_\gamma \binom{\alpha}{\gamma} \frac{\zeta_\gamma(y)}{\zeta_\gamma(1^N)}, \ y \in \mathbb{R}^N.$$

It is known that $\binom{\alpha}{\gamma} = 0$ unless $\gamma^+ \subset \alpha^+$ (that is, $(\gamma^+)_i \le (\alpha^+)_i,\ 1 \le i \le N$). For any scalar s, the homogeneity of ζ_α implies

$$\frac{\zeta_\alpha(y + s1^N)}{\zeta_\alpha(1^N)} = \sum_\gamma \binom{\alpha}{\gamma} s^{|\alpha|-|\gamma|} \frac{\zeta_\gamma(y)}{\zeta_\gamma(1^N)},\ y \in \mathbb{R}^N.$$

When $k = 0$, $\binom{\alpha}{\gamma} = \prod_{i=1}^N \binom{\alpha_i}{\gamma_i}$ (ordinary binomial coefficients).

Proposition 5.8 *(Baker and Forrester [BF3]) For $\alpha \in \mathcal{N}_N$,*

$$\exp\left(s\sum_{i=1}^N y_i\right)\zeta_\alpha(y) = \sum_{\gamma^+ \supset \alpha^+} \frac{h(\alpha^+, k+1)\mathcal{E}_+(\alpha)}{h(\gamma^+, k+1)\mathcal{E}_+(\gamma)} \binom{\gamma}{\alpha} s^{|\gamma|-|\alpha|}\zeta_\gamma(y),\ y \in \mathbb{R}^N,\ s \in \mathbb{R}.$$

Proof. The adjoint in the A-inner product of multiplication by $\exp(s\sum_i y_i)$ is translation $g(y) \mapsto g(y + s1^N)$. Indeed, suppose f and g are polynomials, then

$$\begin{aligned} \langle e^{s\Sigma y_i} f(y), g(y)\rangle_A &= f(T)e^{s\Sigma T_i} g(y)|_{y=0} \\ &= f(T)\exp\left(s\sum_{i=1}^N \frac{\partial}{\partial y_i}\right) g(y)|_{y=0} \\ &= f(T)g(y + s1^N)|_{y=0}, \end{aligned}$$

because $\sum_{i=1}^N T_i = \sum_{i=1}^N \frac{\partial}{\partial y_i}$. The given expression is found by using the A-norms of the orthogonal basis elements ζ_γ (see Corollary 3.19). □

The adjoint of e^{-sL} in the B-product is multiplication by $\exp\left(-s\sum_{i=1}^N x_i^2\right)$, which can be evaluated using the previous result for $\zeta_\gamma(y)$. For $\beta \in \mathcal{N}_N$, $0 \le \ell \le N$, let

$$b(\beta, \ell) = (2\beta_1 + 1, \ldots, 2\beta_\ell + 1, 2\beta_{\ell+1}, \ldots, 2\beta_N).$$

For $\alpha \in \mathcal{N}_N,\ s \in \mathbb{R}$,

$$e^{sL}(x_1 \cdots x_\ell \zeta_\alpha(y)) = \sum_{\beta^+ \subset \alpha^+} \frac{\Lambda(b(\alpha,\ell))h(\beta^+, 1)\mathcal{E}_-(\alpha)}{\Lambda(b(\beta,\ell))h(\alpha^+, 1)\mathcal{E}_-(\beta)} (4s)^{|\alpha|-|\beta|} x_1 x_2 \cdots x_\ell \zeta_\beta(y). \tag{5.7}$$

The formula follows from a similar adjoint-type calculation as in 5.8. Here the norm of the orthogonal basis element is

$$\begin{aligned} \|x_1 x_2 \cdots x_\ell \zeta_\beta(y)\|_B^2 &= 2^{2|\beta|+\ell}\Lambda(b(\beta,\ell))\|\zeta_\beta\|_p^2 \\ &= 2^{2|\beta|+\ell}\Lambda(b(\beta,\ell))h(\beta^+, k+1)\mathcal{E}_+(\beta)\mathcal{E}_-(\beta)/h(\beta^+, 1) \end{aligned}$$

(see (5.4)). Put $s=-\frac{1}{2}$ in the formula to produce an orthogonal basis for $e^{-|x|^2/2}$, $s=-\frac{1}{4}$ for $e^{-|x|^2}$ in the formula (5.5); that is, including all the polynomials $w_A(x_1\cdots x_\ell \zeta_\alpha(y))$, $A\subset [1,N]$, $\ell=\#A$.

The special cases $\ell=0$ and $\ell=N$ have already been obtained by Baker and Forrester [BF3], who called them generalized Laguerre polynomials.

We discuss the connection to the B_N-type spin Calogero model in (1.3).

¿From the commutation $[L,x_i]=2T_i^B$ ([D1], Proposition 2.2), it follows that $e^{-L/2}x_i=(x_i-T_i^B)e^{-L/2}$. Also

$$T_i^B x_i = x_i T_i^B + 1 + 2k_1\sigma_i + k\sum_{j\neq i}(\sigma_{ij}+\tau_{ij}),$$

which shows that

$$e^{-L/2}U_i^B e^{L/2} = x_i T_i^B - (T_i^B)^2 + 1 + k + (2k_1-k)\sigma_i + k\sum_{j>i}(\sigma_{ij}+\tau_{ij}).$$

The Hermite polynomials from (5.6) (with $s=-\frac{1}{2}$) are simultaneous eigenfunctions of these operators. Then

$$\begin{aligned}\mathcal{H}_3 &:= e^{-L/2}\sum_{i=1}^N U_i^B e^{L/2}\\ &= \sum_{i=1}^N x_i\partial_i - \sum_{i=1}^N (T_i^B)^2 + (k_1-k)\sum_{i=1}^N \sigma_i + N(Nk+1+k_1).\end{aligned}$$

The eigenvalues depend only on the degree and the number of odd indices,

$$\mathcal{H}_3 e^{-L/2}(x_1\cdots x_\ell\zeta_\alpha(x^2)) = ((2|\alpha|+\ell)+2\ell(k-k_1)+N(Nk+1+2k_1-k)(e^{-L/2}x_1\cdots x_\ell\zeta_\alpha(x^2)).$$

Let

$$h(x)=\prod_{1\leq i<j\leq N}|x_i^2-x_j^2|^k\prod_{i=1}^N |x_i|^{k_1},$$

then

$$\begin{aligned}h(x)\mathcal{H}_3 h(x)^{-1} &= \sum_{i=1}^N\left(x_i\frac{\partial}{\partial x_i}-\frac{\partial^2}{\partial x_i^2}\right)\\ &\quad + (k_1-k)\sum_{i=1}^N\sigma_i + k_1\sum_{i=1}^N\frac{k_1-\sigma_i}{x_i^2}\\ &\quad + 2k\sum_{1\leq i<j\leq N}\left\{\frac{k-\sigma_{ij}}{(x_i-x_j)^2}+\frac{k-\tau_{ij}}{(x_i+x_j)^2}\right\}+N(k+1).\end{aligned}$$

Conjugating once more leads to

$$e^{-|x|^2/4}h(x)\mathcal{H}_3 h(x)^{-1}e^{|x|^2/4} = \mathcal{H}_2 + (k_1 - k)\sum_{i=1}^{N}\sigma_i + N(k+\frac{1}{2}).$$

The middle term is basically counting the odd indices. Yamamoto [Y] already found that the eigenvalues were evenly spaced. Baker and Forrester [BF3] studied the A-version of this model with a similar transformation, namely $\exp\Big(-\sum_{i=1}^{N} T_i^2/2\Big)$. Van Diejen [vD] and Kakei [K1], [K2], [K3] have studied the symmetric (W_N-invariant) eigenfunctions of this model.

The theory of symmetric and alternating polynomials from Sections 3 and 4 can be applied. We will only write down the two-interval situation, but the methods apply to finer partitions as well.

Fix an interval $[1,\ell]$, let $G_\ell = S_{[1,\ell]} \times S_{[\ell+1,N]}$, and choose $\alpha \in \mathcal{N}_N$ which satisfies $(\geq,[1,\ell])$ and $(\geq,[\ell+1,N])$ (corresponding to a standard B-partition). Then

$$\alpha^R := \sigma_{[1,\ell]}\sigma_{[\ell+1,N]}\alpha = (\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1, \alpha_N, \ldots, \alpha_{\ell+1}),$$

and $\#G_\ell\alpha$ is the number of distinct permutations of $(\alpha_1,\ldots,\alpha_\ell),(\alpha_{\ell+1},\ldots,\alpha_N)$. The following polynomial is an invariant of $S_{[1,\ell]} \times W_{[\ell+1,N]}$: Let

$$\begin{aligned} j_{\alpha;\ell} \;&:=\; x_1x_2\cdots x_\ell\mathcal{E}_+(\alpha^R,[1,\ell])\mathcal{E}_+(\alpha^R,[\ell+1,N]) \\ &\quad\cdot \sum_{\beta\in G_\ell\alpha}\frac{1}{\mathcal{E}_+(\beta,[1,\ell])\mathcal{E}_+(\beta,[\ell+1,N])}\;\zeta_\beta(x_1^2,\ldots,x_N^2). \end{aligned}$$

Then $j_{\alpha;\ell} = x_1x_2\cdots x_\ell\sum_w w\zeta_\alpha$ (summing over a complete set of representatives for the cosets $w\{w_0 \in G_\ell :\ w_0\alpha = \alpha\}$. Further,

$$\|j_{\alpha;\ell}\|_p^2 = (\#G_\ell\alpha)\mathcal{E}_+(\alpha^R,[1,\ell])\mathcal{E}_+(\alpha^R,[\ell+1,N])\|\zeta_\alpha\|_p^2,$$

where

$$\|\zeta_\alpha\|_p^2 = \mathcal{E}_+(\alpha)\mathcal{E}_-(\alpha)h(\alpha^+,k+1)/h(\alpha^+,1);$$

and

$$\|j_{\alpha;\ell}\|_B^2 = 2^{2|\alpha|+\ell}\Lambda(b(\alpha,\ell))\|j_{\alpha;\ell}\|_p^2.$$

This is also the squared norm (from (5.5)) of $e^{-L/2}j_{\alpha;\ell}$, which has the same $S_{[1,\ell]} \times W_{[\ell+1,N]}$ invariance. (For an interval I, W_I is the group generated by S_I and $\{\sigma_i :\ i\in I\}$.)

Further $j_{\alpha;\ell}(1^N) = (\#G_\ell\alpha)\mathcal{E}_-(\alpha)(Nk+1)_{\alpha^+}/h(\alpha^+,1)$.

Suppose that α satisfies $(>,[1,\ell])$ and $(>,[\ell+1,N])$. The following polynomial is alternating for $W_{[1,\ell]} \times S_{[\ell+1,N]}$. Let

$$\begin{aligned} a_{\alpha;\ell} \;&=\; x_1\cdots x_\ell\mathcal{E}_-(\alpha^R,[1,\ell])\mathcal{E}_-(\alpha^R,[\ell+1,N]) \\ &\quad\cdot \sum_{w\in G_\ell}\frac{\operatorname{sgn}(w)}{\mathcal{E}_-(w\alpha,[1,\ell])\mathcal{E}_-(w\alpha,[\ell+1,N])}\;\zeta_{w\alpha}(x_1^2,\ldots,x_N^2). \end{aligned}$$

Then $a_{\alpha;\ell} = x_1 \dots x_\ell \sum_{w \in G_\ell} \operatorname{sgn}(w) w \zeta_\alpha$,

$$\|a_{\alpha;\ell}\|_p^2 = \ell!(N-\ell)!\mathcal{E}_-(\alpha^R, [1,\ell])\mathcal{E}_-(\alpha^R, [\ell+1,N])\|\zeta_\alpha\|_p^2,$$

and

$$\|a_{\alpha;\ell}\|_B^2 = 2^{2|\alpha|+\ell}\Lambda(b(\alpha,\ell))\|a_{\alpha;\ell}\|_p^2.$$

Further,

$$\begin{aligned} & a_{\alpha;\ell}(x)/(a_{[1,\ell]}(x^2)a_{[\ell+1,N]}(x^2))\Big|_{x=1^N} \\ = \; & \frac{(Nk+1)_{\alpha^+}\mathcal{E}_-(\alpha)}{(Nk+1)_\mu h(\alpha^+,1)} \prod \{\kappa_i(\alpha) - \kappa_j(\alpha) - k : \; 1 \le i, j \le \ell \text{ or } \ell+1 \le i < j \le N\}, \end{aligned}$$

where $\mu = (\ell-1, \ell-2, \dots, 1, 0, N-\ell-1, \dots, 1, 0)^+$; recall $a_{[1,\ell]}(x^2) = \prod_{1 \le i < j \le \ell}(x_i^2 - x_j^2)$. Again $e^{-L/2}a_{\alpha;\ell}$ is also alternating for $W_{1,\ell]} \times S_{[\ell+1,N]}$, and its squared norm (for (5.5)) equals $\|a_{\alpha;\ell}\|_B^2$.

As mentioned above, the techniques of sections 2 and 3 can be used to describe polynomials with prescribed symmetry for direct products $W_{I_1} \times W_{I_2} \cdots \times W_{I_r} \times S_{I_{r+1}} \cdots \times S_{I_t}$ for any collection $\{I_1, I_2, \dots, I_t\}$ of pairwise disjoint intervals in $[1, N]$.

More information about the generalized binomial coefficients needs to be obtained so that more concrete algorithms for the type-B Hermite polynomials can be found. The polynomials could be useful in the numerical cubature associated to the Macdonald-Mehta-Selberg integral. Also, such knowledge could lead to orthogonal bases for harmonic polynomials, which, by definition, are annihilated by $\sum_{i=1}^N (T_i^B)^2$; for example, these polynomials appear when one uses spherical polar coordinate systems to find eigenfunctions of some Hamiltonians, see Section 3.4 in [vD].

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